(G)VAR: When is the G essential?

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Abstract

The question of this paper is whether the VAR methodology is able to capture the dynamic and stochastic structure implied by the presence of spillovers in a GVAR system of heterogeneous interacting units. We derive analytical restrictions, propose a test indicating the degree to which aggregation is feasible and derive the asymptotic properties of the VAR estimator in the presence of spillovers. We illustrate the effects of misspecification through a set of numerical examples and find that failing to account for spillovers can lead the econometrician to mistakenly perceive higher persistence and volatility or recover shocks where there are none. Moreover, it is shown that an infinite order VAR can match the dynamics of a first order GVAR. In an empirical application with data for the Euro area we finally show that the recovery of shocks is different both for the forecast error variance decomposition as well as for the historical decomposition, with the VAR placing much more emphasis on its own variables' shocks.

JEL codes: C32, C33, C52, C53,

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1 Introduction

Since the seminal paper of Sims (1980) Vector Auroregression Models (VARs) have been the favorite time series tool in econometrics. The step from univariate to multivariate time series econometrics was essential for modern empirical analysis in economics as many variables are obviously not only influenced by their past realisations but also other variables and their history. Over the years many extensions and modifications were added or suggested. Working with prior knowledge of some parameters has given rise to Bayesian VARs (BVARs), as pioneered by Litterman (1986) and Doan et al. (1984). In addition, the insight that the economy is not fully captured by linear relationships has led to threshold VARs as used by Tsay (1998) or time varying VARs in Amisano and Serati (2003). The more data is available, the more the econometrician wants to use, therefore factor augmented VARs (FAVARs) have been introduced by Bernanke et al. (2004). Some combinations of those exist as well, as the Bayesian FAVAR, exemplified by Ahmadi (2009), with the list of possible VAR specifications continuously expanding.

Recently a new form has been introduced by Pesaran et al. (2004): the Global Autoregression Model (GVAR). The essence of the framework is a VARX where a VAR is estimated for every country (or *entity*, to be more precise) with a weakly exogenous variable included as the weighted average of all other countries/entities. After estimating this VARX for every entity, one can stack and relate the variables and their respective estimated parameter values through appropriate weights. This new tool was especially developed for multi-country analysis, in order to account for spillovers between the countries. Depending on the nature of the spillovers under concern, countries could be linked through financial weights or trade weights or, for different sets of variables, through a combination thereof. Lately GVARs have been used to address various questions. Financial spillovers from the US to other countries have been analyzed by Sgherri and Galesi (2009), spillovers between the housing sectors of the euro area states by Vansteenkiste and Hiebert (2009), between labor markets across the US states by Hiebert and Vansteenkiste (2007). Finally, contributions analyzing fiscal spillovers and monetary policy transmission within the Euro area, for example Hollmayr (2011) and Hebous and Zimmermann (2011), have all benefitted from that new approach.

One might ask where the advantages of the GVAR lie with respect to the VAR besides being able to use individual entity-specific data, capture interlinkages and more countries or sectors. Secondly, it is of interest whether the VAR with aggregated data will nevertheless imply similar dynamics as the GVAR with disaggregated data, thus making little difference for example in terms of forecasting or policy analysis. Put differently, is the disaggregated perspective better able to capture economic dynamics than the macro perspective and is aggregating from the micro to the macro level innocuous? Aggregation and switching from the micro to the macro level is a huge issue in modern macroeconomics and is at the heart of many model critics of the representative agent assumption, for example. However, in empirical exercises this concern has been raised rarely [1], since it is often not possible or incovenient to resort to disaggregated data. The current paper tries to answer the question of whether aggregation may be problematic in certain conditions, by comparing the VAR with the GVAR approach and examining when the two yield the same dynamic responses. Considering the pervasive use of the VAR methodology across the macroeconomic spectrum, from fiscal and monetary policy to banking stress tests, forecasting and financial stability assessments, we regard this question as particularly acute in an increasingly interlinked global economy.

In order to compare the two cases, we start with the simplest possible GVAR model, i.e. two entitites and two variables, and derive conditions when the two approaches yield the same results in terms of the dynamic impulse responses. Starting from these conditions we develop a dynamic equivalence test, by which one can decide if an aggregate VAR captures also the interlinkages and thus aggregation is possible without distorting results. As an empirical illustration we use GDP, inflation and government spending data for all country pairs of the Euro Area and use the proposed testing framework to decide which ones can be aggregated up. As becomes clear in the analytical exercise, aggregation becomes less innocuous, the more entities (and variables) one wants to aggregate up.

In a next step, we calculate the effects of the particular type of misspecification in which the econometrician uses a VAR of aggregated data, with the true data generating process being of a GVAR form, with interlinkages across entities. Following this analytical and empirical

[1] for another recent and similar study with factor augmented VARs confer to Chudik and Pesaran (2009)

analysis, we provide a set of numerical examples and show how impulse responses may change depending on which approach one uses. Furthermore, we illustrate how the econometrician can be misled into recovering shocks to variables that have not experienced shocks, as a consequence of not properly accounting for spillovers.

Moreover we show how in the univariate case the GAR of lag order one can be approximated by an AR(∞), which we attribute to the presence of interlinkages between the entities. The implication of this finding is that, given data limitations in most empirical studies, looking for disaggregated data (at state or industry level) and allowing for spillovers one can reduce the lag order considerably. Chudik and Pesaran (2009) already showed that an infinite VAR (IVAR) can be approximated by a GVAR, however the IVAR relates to the infinity in variables and not the lag structure. The VAR can naturally always be nested in a GVAR, our ideas are therefore not of a normative type.

Finally in an empirical exercise of the Euro Area we compare how different shocks are recovered from a VAR and the GVAR on the same data. Therefore we perform a historical decomposition and a forecast error variance decomposition and find that the GVAR recovers in general more shocks over time, and due to the spillovers the shocks do depend on itself in such big proportions in the forecasting but are more evenly distributed.

2 Analytical results

2.1 Dynamic analysis

The most general set up for the **Global Vector Autoregression** model (GVAR) for a set of N countries/sectors[2] etc. is given in equation (1). At an entity level, this specification is thus of a simple $VARX^*$ form. The methodological innovation of the GVAR consists in the procedure by which the entities are assumed to be related to each other through appropriate

[2] In the rest of the paper we stick to the label entity, as it is possible to extend the GVAR framework beyond the pure country analysis and study sectors, markets etc. Keep in mind that also countries are composed of states or regions.

weighting.

$$\mathbf{x}_{i,t} = \mathbf{c}_i + \mathbf{d}_i t + \sum_{l=1}^{p} \Phi_{il}\mathbf{x}_{i,t-l} + \sum_{l=0}^{q} \Lambda_{il}\mathbf{x}^*_{i,t-l} + \sum_{l=0}^{r} \Psi_{il}\mathbf{f}_{t-l} + \mathbf{u}_{i,t} \tag{1}$$

$$\text{where } \mathbf{u}_{i,t} \sim N(0, \Sigma_i) \text{ and } \mathbf{x}^*_{i,t} = \sum_{j=1}^{N} \tau_{ij}\mathbf{x}_{j,t}$$

We index the entities by $i = 1, \ldots, N$, with $\mathbf{x}_i$ being the vector of endogenous variables. These are regressed on an intercept, a trend and on their lags up to the order p, the contemporaneous and lagged (up to the order q) foreign variables $\mathbf{x}^*_{i,t}$ that represent a weighted average of all others entities' variables with the weights given by $\{\tau_{i,j}\}_{i,j=1,\ldots,N}$ and some common global[3] factors f (also contemporaneous and lagged up to the order r). The error term $\mathbf{u}_i$ is assumed to be normally distributed with mean zero and the variance-covariance matrix Σ_i.

In order to derive analytically tractable results, we abandon this very general setup for the rest of the paper (without loss of generality) and concentrate on a nested version of the model. All results that we derive in the following would of course still apply[4] to the $VARX^*$ model above. In the nested version we consider only two entities and two variables, neither intercept and trend, no common factor and one time lag for both the endogenous variable as well as the foreign one. Furthermore we are also neglecting the contemporaneous spillovers from the rest of the entities. All in all, we allow the two endogenous variables of one entity only to be affected by their own lags as well as by the lags of the other entity's variables. This setup also allows us to neglect the weights between the entities. No matter if one favors trade or financial weights or another weighting scheme, the weighting matrices for both entities thus have the form:

$$W_1 \equiv \begin{bmatrix} \tau_{12} & 0 \\ 0 & \tau_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } W_2 \equiv \begin{bmatrix} 0 & \tau_{21} \\ \tau_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

[3] In most GVAR studies the oil price serves as such a global factor.

[4] Since the full model implies even more complex dynamics than our simple restricted version, our findings can be seen as a lower bound and extending the model can be expected to limit the applicability of VAR even further.

The complete structure that we use is then summarized by the system of equations:

$$\begin{bmatrix} x_{1,1t} \\ x_{1,2t} \end{bmatrix} = \begin{bmatrix} \Phi_{1,11} & \Phi_{1,12} \\ \Phi_{1,21} & \Phi_{1,22} \end{bmatrix} \begin{bmatrix} x_{1,1t-1} \\ x_{1,2t-1} \end{bmatrix} + \begin{bmatrix} \Lambda_{1,11} & \Lambda_{1,12} \\ \Lambda_{1,21} & \Lambda_{1,22} \end{bmatrix} \begin{bmatrix} x_{2,1t-1} \\ x_{2,2t-1} \end{bmatrix} + \begin{bmatrix} u_{1,1t} \\ u_{1,2t} \end{bmatrix}$$

$$\begin{bmatrix} x_{2,1t} \\ x_{2,2t} \end{bmatrix} = \begin{bmatrix} \Phi_{2,11} & \Phi_{2,12} \\ \Phi_{2,21} & \Phi_{2,22} \end{bmatrix} \begin{bmatrix} x_{2,1t-1} \\ x_{2,2t-1} \end{bmatrix} + \begin{bmatrix} \Lambda_{2,11} & \Lambda_{2,12} \\ \Lambda_{2,21} & \Lambda_{2,22} \end{bmatrix} \begin{bmatrix} x_{1,1t-1} \\ x_{1,2t-1} \end{bmatrix} + \begin{bmatrix} u_{2,1t} \\ u_{2,2t} \end{bmatrix}$$

Given the weights, we can stack the two entities underneath each other and arrive at a global solution of the model:

$$\begin{bmatrix} x_{1,1t} \\ x_{1,2t} \\ x_{2,1t} \\ x_{2,2t} \end{bmatrix} = \begin{bmatrix} \Phi_{1,11} & \Phi_{1,12} & \lambda_{1,11} & \lambda_{1,12} \\ \Phi_{1,21} & \Phi_{1,22} & \lambda_{1,21} & \lambda_{1,22} \\ \lambda_{2,11} & \lambda_{2,12} & \Phi_{2,11} & \Phi_{2,12} \\ \lambda_{2,21} & \lambda_{2,22} & \Phi_{2,21} & \Phi_{2,22} \end{bmatrix} \begin{bmatrix} x_{1,1t-1} \\ x_{1,2t-1} \\ x_{2,1t-1} \\ x_{2,2t-1} \end{bmatrix} + \begin{bmatrix} u_{1,1t-1} \\ u_{1,2t-1} \\ u_{2,1t-1} \\ u_{2,2t-1} \end{bmatrix}$$

In order to compare the GVAR approach with the VAR we take the respective variables in every entity and aggregate them to yield a weighted average $\bar{x}_{1,t}$. The aggregation scheme is given by ω and $1-\omega$, i.e. by the relative sizes of the entities. As an example one might think of how big the country or industry is compared to the aggregate. Both aggregate variables can then be written like this:

$$\bar{x}_{1,t} = \omega x_{1,1t} + (1-\omega) x_{2,1t}$$
$$\bar{x}_{2,t} = \omega x_{1,2t} + (1-\omega) x_{2,2t}$$

By the same token the aggregated shocks are given by:

$$\bar{u}_{1,t} = \omega u_{1,1t} + (1-\omega) u_{2,1t}$$
$$\bar{u}_{2,t} = \omega u_{1,2t} + (1-\omega) u_{2,2t}$$

Given these definitions, we specify a VAR of aggregated variables:

$$\begin{bmatrix} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_{1,t-1} \\ \bar{x}_{2,t-1} \end{bmatrix} + \begin{bmatrix} \bar{u}_{1,t-1} \\ \bar{u}_{2,t-1} \end{bmatrix}$$

With the exact distinction of both approaches at hand we proceed now as following. Firstly, we are interested in the case when both models yield precisely the same forecasts or impulse responses. We define this situation as *dynamic equivalence*. More formally, the models are said to be dynamically equivalent if there exists a mapping M such that the coefficient matrix G can be mapped onto the coefficient matrix Ω

$$\underbrace{\begin{bmatrix} \Phi_{1,11} & \Phi_{1,12} & \lambda_{1,11} & \lambda_{1,12} \\ \Phi_{1,21} & \Phi_{1,22} & \lambda_{1,21} & \lambda_{1,22} \\ \lambda_{2,11} & \lambda_{2,12} & \Phi_{2,11} & \Phi_{2,12} \\ \lambda_{2,21} & \lambda_{2,22} & \Phi_{2,21} & \Phi_{2,22} \end{bmatrix}}_{G} \overset{M}{\rightarrow} \underbrace{\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}}_{\Omega}$$

such that:

$$\bar{x}_{1,t|t-1} = \omega x_{1,1t|t-1} + (1-\omega) x_{2,1t|t-1}$$
$$\bar{x}_{2,t|t-1} = \omega x_{1,2t|t-1} + (1-\omega) x_{2,2t|t-1}$$

The VAR-implied dynamics are then for the first variable:

$$\begin{aligned} \bar{x}_{1,t|t-1} &= \Omega_{11}\bar{x}_{1,t-1} + \Omega_{12}\bar{x}_{2,t-1} \\ &= \Omega_{11}[\omega x_{1,1t-1} + (1-\omega) x_{2,1t-1}] + \Omega_{12}[\omega x_{1,2t-1} + (1-\omega) x_{2,2t-1}] \\ &= \omega\Omega_{11} x_{1,1t-1} + (1-\omega)\Omega_{11} x_{2,1t-1} + \omega\Omega_{12} x_{1,2t-1} + (1-\omega)\Omega_{12} x_{2,2t-1} \end{aligned}$$

On the other hand, the GVAR-implied dynamics for the first variable are given by:

$$\begin{aligned} &\omega x_{1,1t|t-1} + (1-\omega) x_{2,1t|t-1} = \\ &\quad = [\omega\Phi_{1,11} + (1-\omega)\lambda_{2,11}] x_{1,1t-1} + [\omega\lambda_{1,11} + (1-\omega)\Phi_{2,11}] x_{2,1t-1} + \\ &\quad [\omega\Phi_{1,12} + (1-\omega)\lambda_{2,12}] x_{1,2t-1} + [\omega\lambda_{1,12} + (1-\omega)\Phi_{2,12}] x_{2,2t-1} \end{aligned}$$

The following restrictions[5] can then be derived:

$$\Omega_{11} = \Phi_{1,11} + \frac{1-\omega}{\omega}\lambda_{2,11} \stackrel{!}{=} \Phi_{2,11} + \frac{\omega}{1-\omega}\lambda_{1,11}$$
$$\Omega_{12} = \Phi_{1,12} + \frac{1-\omega}{\omega}\lambda_{2,12} \stackrel{!}{=} \Phi_{2,12} + \frac{\omega}{1-\omega}\lambda_{1,12}$$

These restrictions can be interpreted as saying that the first variable's dependence on its lag in the aggregated VAR case must be equivalent to the effect of the lag of the disaggregated first variable of the first entity plus the spillover from the other entity's first variable's lag and also the first variable of the second entity plus the reverse weighting scheme from the first entity. This has to hold for all four coefficients in the coefficient matrix of Ω. If we neglect all spillovers, i.e. if we set all λs to zero, we are in the simple case of aggregating VARs and the second terms of weighted spillovers would drop.

If we increase the number of entities the restrictions become harder to satisfy as more coefficients need to be identical at the same time. The weight of the specific spillover from another entity would diminish, however, as the weight of this particular entity is now less than in the two-entity example. In short, the more entities one wants to aggregate, the less probable it is that those conditions hold, i.e. it becomes less innocuous to aggregate, especially if one considers very different entities.

Although these are analytical restrictions which insure that both approaches yield the exact identical impulse response functions, in reality both might nevertheless be very similar and economically indistinguishable. Therefore we propose an F-test for the current case of two entities and two variables to decide whether aggregation is innocuous or not. The econometric restrictions are then given by the linear F-test:

$$R \text{ vec } G \stackrel{!}{=} 0$$

[5] The restrictions for the other variables are straightforward and are presented in detail in the Appendix.

with:

$$R = \begin{bmatrix} R_1 & 0_{2\times4} & R_2 & 0_{2\times4} \\ 0_{2\times4} & R_1 & 0_{2\times4} & R_2 \end{bmatrix}$$

where:

$$R_1 = \begin{bmatrix} 1 & 0 & \frac{1-\omega}{\omega} & 0 \\ 0 & 1 & 0 & \frac{1-\omega}{\omega} \end{bmatrix} \text{ and } R_2 = \begin{bmatrix} -\frac{\omega}{1-\omega} & 0 & -1 & 0 \\ 0 & -\frac{\omega}{1-\omega} & 0 & -1 \end{bmatrix}$$

The generalization of this F-test to N entities and K variables can be found in the Appendix. It is structured in the same way as the two-dimensional one but now also with trade weights τ_{ij} and aggregation weights $\frac{\omega_i}{\omega_j}$ being present. It works exactly as in the simple framework with entering the estimated coefficients and calibrated trade weights at the right positions in the matrix R and comparing it via the dynamic equivalence test to the matrix G where the VAR estimated coefficients are stored. To conclude this derivation of the restrictions and the development of the test, we apply it to Euro area data and use the new test for dynamic equivalence. For all country pairs of the Euro Area original member countries we try to aggregate once GDP and inflation and then GDP and government spending (results are reported in Tables 1 and 2 in the Appendix). Based on the results of the F-test we conclude that there is strong evidence against the aggregation of Spain with all other Euro area countries, while there seems to be no evidence against bilateral aggregation of small and rather economically similar countries like Belgium, the Netherlands and Austria.

2.2 Effects of misspecification

Assume we are in a world nested by the specification in (1), with a set of N entities, the data generating process for the sets of $k_i, i = 1, \ldots, N$ variables being of a $VARX^*$ form:

$$\mathbf{x}_{i,t} = \Phi_i \mathbf{x}_{i,t-1} + \Lambda_i \mathbf{x}^*_{i,t-1} + \mathbf{u}_{i,t}$$
$$\text{where } \mathbf{u}_{i,t} \sim N(0, \Sigma_i) \text{ and } \mathbf{x}^*_{i,t} = \sum_{j=1}^{N} \tau_{ij} \mathbf{x}_{j,t}$$

We define the stacked variables:

$$\mathbf{x}_t = \begin{bmatrix} \mathbf{x}_{1,t} \\ \mathbf{x}_{2,t} \\ \vdots \\ \mathbf{x}_{N,t} \end{bmatrix} \text{ and } \mathbf{u}_t = \begin{bmatrix} \mathbf{u}_{1,t} \\ \mathbf{u}_{2,t} \\ \vdots \\ \mathbf{u}_{N,t} \end{bmatrix}$$

such that we then have $\mathbf{x}^*_{i,t} = W_i \mathbf{x}_t$, where W_i is the weighting matrix with the non-zero elements given by $\{\tau_{ij}\}_{i,j=1,\ldots,N}$.

The system can then be written in stacked form:

$$\mathbf{x}_t = G\mathbf{x}_{t-1} + \mathbf{u}_t, \quad \mathbf{u}_t \sim N(0, \Sigma) \tag{2}$$

where G is a function of $\{\Phi_i, \Lambda_i, W_i\}_{i=1,\ldots,N}$. For details of this derivation we refer the reader to Pesaran et al. (2004).

Assume now that what the econometrician observes are aggregated variables:

$$\bar{\mathbf{x}}_t = W\mathbf{x}_t, \text{ with } W = \begin{bmatrix} \omega_1 & 0 & \cdots & \omega_2 & 0 & \cdots & \omega_N & 0 & \cdots \\ 0 & \omega_1 & \cdots & 0 & \omega_2 & \cdots & 0 & \omega_N & \cdots \\ \cdots & & & & & & & & \end{bmatrix}.$$

The estimation that the econometrician is then carrying out is:

$$\bar{\mathbf{x}}_t = \hat{\Omega}\bar{\mathbf{x}}_{t-1} + \hat{\mathbf{e}}_t, \quad \hat{\mathbf{e}}_t \sim N(0, \hat{\Theta}) \quad \text{with} \quad \bar{\mathbf{x}}_t = W\mathbf{x}_t \tag{3}$$

We now state in the following Proposition our main results.

Proposition 1. *If a system of N units is driven by individual $VARX^*$ processes with an arbitrary structure of spillovers, as captured by equation* (2)*, the econometrician who has access to only aggregated variables and thus estimates an aggregated VAR system of the form* (3)*, the estimated coefficients will relate to actual ones according to the equation:*

$$\hat{\Omega} = (W'\Gamma_0 W)^{-1} W'\Gamma_0 G W, \tag{4}$$

where Γ_0 *is given by the matrix equation* $\Gamma_0 = G'\Gamma_0 G + \Sigma$.

In addition, the variance-covariance matrix of aggregated shocks will be estimated as:

$$\hat{\Theta} = E[\hat{\mathbf{e}}_t'\hat{\mathbf{e}}_t] = W'\Sigma W + \Delta'\Gamma_0\Delta \tag{5}$$

where $\Delta = \left(I_{N\times\sum_{i=1}^{N}k_i} - W(W'\Gamma_0 W)^{-1}W'\Gamma_0\right)GW$.

Finally, the aggregated shocks will be recovered as:

$$\hat{\mathbf{e}}_t = \mathbf{u}_t W + \bar{\mathbf{x}}_{t-1}\Delta$$

Proof. We provide derivations in the corresponding section of the Appendix. □

The main message of Proposition 1 concerns the recovery of shocks and the estimated variance-covariance matrix at an aggregate level. The matrix Δ is key in this regard. If $\Delta = 0$, then the shocks are correctly recovered as simple weighted averages of true shocks, while the variance-covariance matrix is correctly estimated, as an appropriately weighted version of the unit-specific variance-covariance matrix. However, as is immediately apparent, if $\Delta \neq 0$, the aggregated VAR is not able to replicate the stochastic structure affecting the economy and propagated by the underlying system of $VARX^*$ models.

3 Numerical results

3.1 Univariate case

The main intuition for our results can be illustrated in a basic univariate setup. We start by assuming that the data generating process is a GAR(1) model:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} \phi_1 & \lambda_1 \\ \lambda_2 & \phi_2 \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}, \text{ where } \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

Aggregation implies:

$$\bar{x}_t = \omega x_{1,t} + (1-\omega)x_{2,t}$$

The econometrician then estimates:

$$\bar{x}_t = \hat{\phi}\bar{x}_{t-1} + \hat{e}_t$$

We now consider a set of possible cases, which we summarize below, with the simulation results being reported in Figure 1. We depict impulse responses to an aggregate shock, i.e. a shock originating proportionally in both countries. In the first case we consider a simple system of two identical countries with no spillovers. The estimated coefficient is naturally exactly identical to the true coefficients and the impulse responses are identical to their actual counterparts. In Case 2, when the two countries differ with respect to the persistence parameter, it is already visible that although the estimated aggregate coefficient is trying to match the persistence of the aggregated variable, the impulse response is deviating from the truth. This happens simply because any weighted sum of the impulse responses cannot be equal to the impulse response of the weighted sum of the variables.

Calibration					**Estimation**
	ϕ_1	ϕ_2	λ_1	λ_2	$\hat{\phi}$
1:	0.5	0.5	0	0	0.5
2:	0.9	0.1	0	0	0.77
3:	0.5	0.5	1.5	0	0.83
4:	0.5	0.5	-1.5	0	0.5
5:	0.9	0.1	1	-0.5	0.56

In Case 3 we observe that the presence of the spillover induces additional persistence in the estimated process for the aggregate variable and the estimated aggregate impulse response deviates from the true one despite the variable from both countries actually having the same persistence. Case 4 is the counterpart thereto, but with a negative spillover and case 5 depicts the most general case possible in this context, with differences in persistence and both negative and positive spillovers across countries. The main conclusion is that the dynamics captured by the VAR can be very different from the true one, not only in quantitative terms, but also magnitude-wise.

In Figure 2 we show (in the context of the same calibration as in case 5) that in the presence of spillovers the dynamics inherent in a GAR model can be replicated by an infinite-lag AR setup. The more lags one allows for, the closer the AR process can capture the dynamics of the GAR, because feedback effects between the entities go back to $t = 0$, therefore the aggregated AR process has to be as flexible as possible to match the GAR dynamics.

3.2 Multivariate case

We now work in the multivariate case analyzed in Section 2, with the notation corresponding thereto. We consider two cases, the first one being the case of spillovers across identical variables.

The stacked representation of the model is:

$$\begin{bmatrix} x_{1,1t} \\ x_{1,2t} \\ x_{2,1t} \\ x_{2,2t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 1 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}}_{G} \begin{bmatrix} x_{1,1t-1} \\ x_{1,2t-1} \\ x_{2,1t-1} \\ x_{2,2t-1} \end{bmatrix} + \begin{bmatrix} u_{1,1t-1} \\ u_{1,2t-1} \\ u_{2,1t-1} \\ u_{2,2t-1} \end{bmatrix}$$

For the stochastic structure we assume:

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \omega = 0.5 \Rightarrow \Theta = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

The asymptotic estimators of the econometrician are then:

$$\hat{\Omega} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix} \text{ and } \hat{\Theta} = \begin{bmatrix} 0.6667 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Two results stand out: the first variable appears to be much more persistent than in reality and the variance of the first shock is overestimated by a factor of around 20%. To further build up intuition, we depict in Figure 3 the evolution of the variables through time, in response to a shock to the first variable. We observe an immediate response of the first variable in the first country and a lagged one in the second country. Also, the shock that the VAR assigns to the first period is much higher (0.8 compared to 0.5) than the actual one, because the VAR includes both the direct effect of the shock, as well as the spillover. After the initial period, the VAR even recovers a series of negative shocks, in an attempt to correct for the initial overstatement. Figure 4 illustrates this observation through a visualization of recovered aggregate shocks under a stochastic time sequence of shocks to the first variable continuously affecting the system.

The presence of spillovers from the first country to the second generates a hump-shaped impulse response in the second country and also at aggregate level, because the exogenous persistence of the variable and the endogenous persistence of the system are at work simultaneously. We observe hump-shaped impulse responses throughout macroeconometrics and usually they are explained from the theoretical perspective by resorting to market frictions, so our result may be relevant as an attempt to an additional - systemic - explanation of the gradual diffusion of shocks throughout the economy.

The second experiment assumes the following stacked representation of the model:

$$\begin{bmatrix} x_{1,1t} \\ x_{1,2t} \\ x_{2,1t} \\ x_{2,2t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ -1 & 0 & 0 & 0.5 \end{bmatrix}}_{G} \begin{bmatrix} x_{1,1t-1} \\ x_{1,2t-1} \\ x_{2,1t-1} \\ x_{2,2t-1} \end{bmatrix} + \begin{bmatrix} u_{1,1t-1} \\ u_{1,2t-1} \\ u_{2,1t-1} \\ u_{2,2t-1} \end{bmatrix}$$

The stochastic structure is:

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \omega = 0.5 \Rightarrow \Theta = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

And the asymptotic estimators are:

$$\hat{\Omega} = \begin{bmatrix} 0.5 & 0 \\ -0.4722 & 0.5833 \end{bmatrix} \text{ and } \hat{\Theta} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.6574 \end{bmatrix}$$

We depict in Figure 3 the evolution of the variables through time, in response to a shock to the first variable. The conclusions are similar to the previous experiment, with one notable difference: the spillovers are now assumed to go from the first variable in the first country to the second variable of the second country. This again generates hump-shaped responses at aggregate level, but we also observe that the VAR recovers in the initial period an aggregated shock to the second variable. The simulated time series in Figure 5 additionally illustrate this point, by indicating that, in the presence of cross-unit and cross-variable spillovers, shocks are identified where actually there are none.

4 Empirical Application

In order to illustrate empirically what we have shown in chapter 3 in a numerical example with just two entities and two variables we document the differences between the two approaches in recovering shocks from a VAR on aggregate data of the Euro Area (EMU11) and the GVAR with disaggregated data on the 11 individual countries.

4.1 Data

The largest part of the data stems from the OECD quarterly data set and comprises three variables, real GDP, real government spending and the CPI index for the eleven original Euro Area member countries [6]. All of them go back until $1971Q1$ and last until $2009Q4$. Taking first differences on a year-on-year basis we sacrifice the first year 1971 and thereby transform the variables into growth rates for GDP and government spending and inflation. The fourth variable is the common short run nominal interest rate that stems from the New Area Wide Data set of the European Central Bank (ECB). Its frequency is also quarterly and the time horizon is the same as for the other three variable. For the time before the beginning of the currency union, the interest rate is artificially constructed for the member countries. It is the same variable that for example Smets and Wouters (2003) use in their model of the Euro Area. The data for the calibrated trade weights (see Table 3 in the appendix) come from the import and export data from each country vis-a-vis the rest of the Euro area countries. The frequency of those data is yearly and covers the period between 1999 and 2008. As the trade weights in this set up are not time varying, a simple average of those ten years is taken. In the aggregation step of both estimation procedures, we choose simple GDP weights (see Table 3 in the appendix) of the eleven countries corresponding to the end of the sample period 2008.

4.2 Models and Estimation

The estimation of the GVAR is carried out on the disaggregated country specific data with the interest rate as the common factor across countries. The estimation for every country i is given by a $VARX*$ multivariate process of the form:

$$x_{i,t} = a_{i,0} + a_{i,1}t + \Phi_{i,1}x_{i,t-1} + \Phi_{i,t}x_{i,t-2} + \Lambda_{i,0}x^*_{i,t} + \Lambda i, 1x^*_{i,t-1} + \Upsilon_{i,0}d_t + \Upsilon_{i,1}d_{t-1} + \epsilon_{i,t} \quad (6)$$

where $x_{i,t} = (Y_{i,t}, G_{i,t}, \pi_{i,t})'$ is a vector of country-specific GDP, government spending and inflation respectively, $x^*_{i,t} = (Y^*_{i,t}, G^*_{i,t}, \pi^*_{i,t})'$ are foreign counterparts thereof and $d_t = R_t$ is

[6]These countries are Austria, Belgium, Germany, Finland, France, Ireland, Italy, Luxembourg, Netherlands, Portugal and Spain

the nominal (common) interest rate. More precisely, foreign GDP for country i is for example given by $Y_{i,t}^{*} = \sum_{j=0}^{N} w_{ij} Y_{j,t}$, i.e. corresponds to a weighted average of all foreign economies' GDP. As the comparison with the area-wide VAR that is estimated later should be as close as possible, we renounce on the possibility to let the data speak for itself fully in terms of optimal lag lengths and instead choose an exogenous lag length of two throughout the exercises. For the sake of parsimony, the lag length for the foreign variables and the common factor are chosen to be one.

The estimation procedures consists of two steps. First, we run a series of country-level $VARX^{*}$ models, to obtain the coefficient matrices and the error terms for each country i. Second, in order to arrive at the global solution of the interconnected system we have to tie the countries together via the trade weights. Hereby, we define the new vector $z_{i,t}$:

$$z_{i,t} = \begin{bmatrix} x_{i,t} \\ x_{i,t}^{*} \end{bmatrix}$$

and transform equation (6) to arrive at:

$$A_i z_{it} = a_{i0} + a_{i1} t + B_{i,0} z_{i,t-1} + +B_{i,1} z_{i,t-2} + \Upsilon_{i,0} d_t + \Upsilon_{i,1} d_{t-1} + +\epsilon_{i,t} \tag{7}$$

with $A_i = (I_K - \Lambda_{i,0})$,$B_{i,1} = (\Phi_{i,1} \Lambda_{i,1})$ and $B_{i,2} = (\Phi_{i,2} 0_{K \times K})$ for each country i.

With the trade matrix W_i generated for every country we can then relate the vector $z_{i,t}$ to the full set of endogenous variables over the set of countries x_t by writing:

$$z_{it} = W_i X_t,$$

to finally get the solution:

$$GX_t = a_0 + a_1 t + HX_{t-1} + \Xi_0 d_t + \Xi_1 d_{t-1} + u_t$$

where $G = \begin{bmatrix} A_1W_1 \\ A_2W_2 \\ ... \\ A_NW_N \end{bmatrix}$,$H = \begin{bmatrix} B_1W_1 \\ B_2W_2 \\ ... \\ B_NW_N \end{bmatrix}$,$\Xi_l = \begin{bmatrix} \Upsilon_{1,l} \\ \Upsilon_{2,l} \\ ... \\ \Upsilon_{N,l} \end{bmatrix}$ and $u_t = \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ ... \\ \epsilon_{N,t} \end{bmatrix}$

The reduced form global solution is given by the inverse of the matrix G. Forming the companion form matrix F we obtain:

$$X_t = FX_{t-1} + J_0 d_t + J_1 d_{t-1} + \varepsilon_t$$

The matrix F is then finally a $(2 + 6 \cdot 11) \times (2 + 6 \cdot 11)$ matrix. As is common in the GVAR literature, see for example Sgherri and Galesi (2009), we use a generalized identification strategy of structural shocks. The reason is that in the orthogonalized strategy one would have to order variables and moreover in the GVAR one would have to take a stance a priori on the cross-border transmission of shocks, with a reasonable ordering being unfounded here.

Parallel to the GVAR, we estimate an area-wide VAR on the same data. We aggregate up the country specific growth rates of GDP, government spending and inflation by the GDP weights. The interest rate is left untouched as it serves as a common factor in the GVAR [7]. Many simple VAR models that look for example at monetary transmission also use data for GDP, inflation and the interest rate, see for example Georgiadis (2011) and Cecioni and Neri (2011). The incorporation of government variables has become more and more standard, though (see e.g. van Aarle et al. (2003)). The estimation equation looks like this:

$$x_t = a_0 + a_1 t + \phi_0 x_{t-1} + \phi_1 x_{t-2} + \epsilon_t \tag{8}$$

where x_t is the vector composed of (Y_t, G_t, π_t, R_t). This VAR is also estimated by least squares and written in companion form, $x_t = f x_{t-1} + \varepsilon_t$ after the coefficient matrices and the error terms have been obtained. As in the case of the GVAR the identification is equivalently chosen to be the generalized one.

Moreover in order to be able to compare both methods as closely as possible and to obtain

[7]Naturally, if it were to be aggregated up, the result would exactly be the same as the original data.

shocks also for the interest rate in the GVAR case, we follow a new route and stack the common factor and its lag beneath the observable variables of all countries. The same has to be done in the coefficient matrix and for the error terms so that the following equation holds: $Y_t = ZY_{t-1} + e_t$ with $Y_t = \begin{bmatrix} X_t \\ d_t \\ d_{t-1} \end{bmatrix}$ and the new companion form matrix is now written as:

$$Z = \begin{bmatrix} F, J_0, J_1 \\ f_{4,1:2}, [w \bigotimes (f_{4,3:5}, f_{4,7:9})], f_{4,6}, f_{4,10} \end{bmatrix}$$

with w being the vector of GDP weights and $f_{4,:}$ corresponding to the last line of the companion form matrix from the VAR case, i.e. a vector of coefficients indicating how the interest rate is affected by the other endogenous variables, split up across countries. The last entries in the error vector also stem from the VAR case, or equivalently the variance of the interest rate in the covariance matrix of the GVAR is taken from the covariance matrix of the VAR. So for the sake of comparability of the different methodologies we gave the VAR a more than fair chance of being identical in terms of the stochastic structure. Finally, there is once again the difference in the amount of shocks and variables for the two methodologies, as the GVAR has 34 variables and shocks (eleven countries, three variables each and the interest rate) whereas the VAR only has four variables and shocks. To compare the two and draw conclusion whether aggregation is innocuous and whether the "G" is essential we aggregate the shocks over the eleven countries up. The aggregation is once again carried out with the GDP weights. So overall differences can only arise because of the fact that the aggregation steps are performed at different stages of the estimation and that in one case spillovers are adequately accounted for.

4.3 Results

After having obtained the coefficient and the covariance matrices of the VAR and the GVAR respectively we perform a Forecast Error Variance Decomposition (FEVD) and a Historical decomposition of shocks and compare both methods with each other. Both decompositions are explained and used in Dees et al. (2010) and Smets and Wouters (2003) respectively [8].

4.3.1 Forecast Error Variance Decomposition

Figures 6 to 9 depict the forecast error variance decomposition for the VAR on the left hand side and for the GVAR on the right hand side for the horizons 5 periods ahead to 40 ahead. In each figure both methods display similar patterns, which is reassuring in terms of analyzing the estimation methods comparatively. For the 5 period ahead forecast horizon for each variable the majority is explained by own shocks. With the exception of the interest rate for all the other variables the own shock is much more important in the VAR case than in the GVAR case. Demand and fiscal shocks are more evenly distributed for all variables. Looking just at real GDP and government expenditure, both seem to depend much more on each other than in the VAR case. This serves as an indication that fiscal spillovers between the countries indeed do exist (see also the studies of Hollmayr (2011) and Hebous and Zimmermann (2011) on fiscal spillovers in the Euro Area). The nominal interest rate is more important for all the other variables in the VAR case, a phenomenon that holds also for longer forecast horizons. The supply shock is bigger both for real GDP and government expenditure, not for the interest rate, however. The whole pattern changes in figures 7 and 8, with the own shock explaining both for the VAR and the GVAR case equal amounts - with the exception of government spending. The fiscal shock once again plays a bigger role for all the other variables in the GVAR framework, suggesting the presence of spillovers. Finally, after 40 periods the differences grow once again, with the monetary policy shock now affecting the nominal interest rate considerably less than before. Inflation in the long run in our sample seems to be more affected by demand shocks (fiscal and GDP) and less by the supply side in the GVAR case than in the VAR case. Smets and Peersman (2001) show also in a VAR analysis for the Euro area in a subsample of our time series that the monetary policy shock is not influencing much both the demand side nor the supply side. This holds true also in our version with the effect being downplayed even more in the GVAR case. This result is also in line with the study of Dees et al. (2010) who perform a GVAR for 33 countries. The forecast error variance decomposition for the euro area concerning the effect of the monetary policy

[8] For an overview of how to compute both decompositions, please refer to those sources.

on the other variables is very much alike and also subdued for most of the forecast horizon. What seems to be the overall result of the comparison between the two approaches, however, is that the self-explanation of one's shock onto each variable is higher in the VAR case for all horizons, suggesting that spillovers do matter and distort the VAR results at least a little.

4.3.2 Historical Decomposition

In the historical decomposition we plot only averages of four quarters, so that in each figure 10 and 11 there are 37 columns, one for each year of the time series. The data which is depicted by the black line must be the same no matter what methodology one uses. Only the four shocks driving the data are somewhat differently composed for many years and variables. The general observation is equivalent to the result before in the forecast error variance decomposition. Comparing both figures one observes that in the VAR case the variables seem to be much more driven by their own shocks in the whole sample period compared to the GVAR case. Altogether the size of the shocks is higher in the GVAR approach. The nominal interest rate is displaying the most similar pattern in terms of shock decomposition. This is perhaps due to the fact of how this variable was integrated in the GVAR methodology for the shock decomposition (coefficients stemming directly from the VAR approach). One major difference is that also the fiscal side contributes to the development of the nominal interest rate in the GVAR whereas in the VAR the fiscal shock is tiny or not present at all. The shocks driving inflation, however, are very different. In the nineties, for example, the predominant negative shock in the VAR model was the supply shock. In the GVAR supply was also negative, but fiscal and also monetary shocks contributed a lot to the negative development as well while demand remained highly positive. This latter shock the VAR just does not recover at all. In a different study using a DSGE model but also decomposing supply and demand shocks for the Euro Area Sahuc and Smets (2006) find that in this period both supply and demand shocks are negative, a finding that would most certainly look different if the authors accounted for disaggregated data. For the growth rate of real GDP fiscal demand is much more essential in the GVAR framework than in the VAR case. Once again, as was the case in 4.3.1, a hint that fiscal spillovers might have been active the whole time. In the years from 2000 onwards the positive growth rate of GDP is also due to monetary shocks, a phenomenon the VAR does not capture. The supply shocks are throughout very small in the VAR case, a finding that corresponds also to the result of Barthlemy et al. (2009) who also use Euro area data in a DSGE model and find that the overall supply shocks are relatively modest compared to the demand shocks explaining this variable. The data for government expenditure in the VAR is completely driven by its own shock, with overall demand at times counteracting or enforcing the leading fiscal shock. In the GVAR the fiscal variable encounters times with huge counteracting fiscal and overall demand shocks and times of very smooth ones.

Note that the theoretical result of the numerical analysis in chapter 3 when the data generating process was a GVAR and the VAR recovered shocks of variables in the VAR was wrong might also be transferred to this empirical exercise. First of all, here the data generating process is not a priori the GVAR (albeit the GVAR has more free parameters and thus fits the data better in-sample, by construction). Then, in contrast to the theoretical exercise we do not know the true shock(s) in the present empirical analysis. But in coherence with chapter 3, we conclude that the overstatement of the own shocks in the VAR case is at least questionable, and the more evenly contribution of all shocks in the GVAR the better alternative given that spillovers between the countries are highly probable.

5 Conclusion

This paper tries to shed some light on the issue of aggregation in applied macroeconometrics. Our main question is whether economic dynamics in a system of multiple heterogeneous units (e.g. like a monetary union with fiscally independent countries) in which spillovers of an arbitrary form exist (e.g. trade links or a coordination mechanism like the single monetary policy of the ECB) can be captured sufficiently well in an aggregated VAR analysis. We first derive explicit restrictions such that the aggregated VAR and the alternative disaggregated GVAR are dynamically equivalent and propose a test to serve as a guidance for the researcher interested in assessing whether the presence of the spillovers is economically relevant.

We find in a simple empirical exercise with actual bilateral Euro area data that most of the time for just two variables and two countries, it can still be argued statistically in favor of spillovers. It can be concluded that the more variables and the more entities one wants to include in the VAR, the harder it is not to have the data reject the imposed restrictions.

In order to gain more intuition as to what the effects of the presence of spillovers are on aggregate dynamics we first provide an analytical derivation of the magnitude of the misspecification that arise once the aggregated VAR is estimated and the true data generating process is of a disaggregated GVAR form. By use of simple numerical simulations we show that the misspecification effects are substantial especially as concerns the recovered shock processes, with shocks appearing as more persistent and more volatile to an econometrician which has just access to aggregated data. Also, since the presence of spillovers gives rise to hump-shaped impulse responses, we think spillovers could provide a rationale for the lagged and persistent response of aggregate variables to fundamental shocks, beyond market frictions.

In an empirical application for the Euro Area we finally show that recovering shocks yields different results with both methodologies. Inflation for example was undergoing positive demand shocks in the nineties if one relies on the GVAR whereas the VAR does not capture this shock in this period at all. In the VAR the response of the own shocks to each variable is much more pronounced than in the GVAR where the effects of the different shocks are more evenly distributed.

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Appendix

A. Derivations and Proofs:

Restrictions for all variables:

$$\begin{aligned}\bar{x}_{1,t} &= \omega x_{1,1t} + (1-\omega)x_{2,1t} \\ &= \omega(\Phi_{1,11}x_{1,1t-1} + \Phi_{1,12}x_{1,2t-1} + \lambda_{1,11}x_{2,1t-1} + \lambda_{1,12}x_{2,2t-1}) + \\ &\quad (1-\omega)(\lambda_{2,11}x_{1,1t-1} + \lambda_{2,12}x_{1,2t-1} + \Phi_{2,11}x_{2,1t-1} + \Phi_{2,12}x_{2,2t-1}) \\ &= [\omega\Phi_{1,11} + (1-\omega)\lambda_{2,11}]\, x_{1,1t-1} + [\omega\lambda_{1,11} + (1-\omega)\Phi_{2,11}]\, x_{2,1t-1} + \\ &\quad [\omega\Phi_{1,12} + (1-\omega)\lambda_{2,12}]\, x_{1,2t-1} + [\omega\lambda_{1,12} + (1-\omega)\Phi_{2,12}]\, x_{2,2t-1}\end{aligned}$$

$$\begin{aligned}\bar{x}_{1,t} &= \Omega_{11}(\omega x_{1,1t-1} + (1-\omega)x_{2,1t-1}) + \Omega_{12}(\omega x_{1,2t-1} + (1-\omega)x_{2,2t-1}) \\ &= \omega\Omega_{11}x_{1,1t-1} + (1-\omega)\Omega_{11}x_{2,1t-1} + \\ &\quad \omega\Omega_{12}x_{1,2t-1} + (1-\omega)\Omega_{12}x_{2,2t-1}\end{aligned}$$

$$\begin{aligned}\omega\Omega_{11} &= \omega\Phi_{1,11} + (1-\omega)\lambda_{2,11} \\ (1-\omega)\Omega_{11} &= \omega\lambda_{1,11} + (1-\omega)\Phi_{2,11}\end{aligned}$$

$$\begin{aligned}\Omega_{11} &= \frac{\omega\Phi_{1,11} + (1-\omega)\lambda_{2,11}}{\omega} = \frac{\omega\lambda_{1,11} + (1-\omega)\Phi_{2,11}}{1-\omega} \\ &= \Phi_{1,11} + \frac{(1-\omega)}{\omega}\lambda_{2,11} = \Phi_{2,11} + \frac{\omega}{1-\omega}\lambda_{1,11}\end{aligned}$$

$$\begin{aligned}\omega\Omega_{12} &= \omega\Phi_{1,12} + (1-\omega)\lambda_{2,12} \\ (1-\omega)\Omega_{12} &= \omega\lambda_{1,12} + (1-\omega)\Phi_{2,12}\end{aligned}$$

$$\Omega_{12} = \frac{\omega\Phi_{1,12} + (1-\omega)\lambda_{2,12}}{\omega} = \frac{\omega\lambda_{1,12} + (1-\omega)\Phi_{2,12}}{1-\omega}$$
$$= \Phi_{1,12} + \frac{(1-\omega)}{\omega}\lambda_{2,12} = \Phi_{2,12} + \frac{\omega}{1-\omega}\lambda_{1,12}$$

$$\begin{aligned}
\bar{x}_{2,t} &= \omega x_{1,2t} + (1-\omega)x_{2,2t} \\
&= \omega(\Phi_{1,21}x_{1,1t-1} + \Phi_{1,22}x_{1,2t-1} + \lambda_{1,21}x_{2,1t-1} + \lambda_{1,22}x_{2,2t-1}) + \\
&\quad (1-\omega)(\lambda_{2,21}x_{1,1t-1} + \lambda_{2,22}x_{1,2t-1} + \Phi_{2,21}x_{2,1t-1} + \Phi_{2,22}x_{2,2t-1}) \\
&= [\omega\Phi_{1,21} + (1-\omega)\lambda_{2,21}]\, x_{1,1t-1} + [\omega\lambda_{1,21} + (1-\omega)\Phi_{2,21}]\, x_{2,1t-1} + \\
&\quad [\omega\Phi_{1,22} + (1-\omega)\lambda_{2,22}]\, x_{1,2t-1} + [\omega\lambda_{1,22} + (1-\omega)\Phi_{2,22}]\, x_{2,2t-1}
\end{aligned}$$

$$\begin{aligned}
\bar{x}_{2,t} &= \Omega_{21}(\omega x_{1,1t-1} + (1-\omega)x_{2,1t-1}) + \Omega_{22}(\omega x_{1,2t-1} + (1-\omega)x_{2,2t-1}) \\
&= \omega\Omega_{21}x_{1,1t-1} + (1-\omega)\Omega_{21}x_{2,1t-1} + \\
&\quad \omega\Omega_{22}x_{1,2t-1} + (1-\omega)\Omega_{22}x_{2,2t-1}
\end{aligned}$$

$$\omega\Omega_{21} = \omega\Phi_{1,21} + (1-\omega)\lambda_{2,21}$$
$$(1-\omega)\Omega_{21} = \omega\lambda_{1,21} + (1-\omega)\Phi_{2,21}$$

$$\Omega_{21} = \frac{\omega\Phi_{1,21} + (1-\omega)\lambda_{2,21}}{\omega} = \frac{\omega\lambda_{1,21} + (1-\omega)\Phi_{2,21}}{1-\omega}$$
$$= \Phi_{1,21} + \frac{(1-\omega)}{\omega}\lambda_{2,11} = \Phi_{2,21} + \frac{\omega}{1-\omega}\lambda_{1,21}$$

$$\omega\Omega_{22} = \omega\Phi_{1,22} + (1-\omega)\lambda_{2,22}$$
$$(1-\omega)\Omega_{22} = \omega\lambda_{1,22} + (1-\omega)\Phi_{2,22}$$

$$\Omega_{22} = \frac{\omega\Phi_{1,22} + (1-\omega)\lambda_{2,22}}{\omega} = \frac{\omega\lambda_{1,22} + (1-\omega)\Phi_{2,22}}{1-\omega}$$
$$= \Phi_{1,22} + \frac{(1-\omega)}{\omega}\lambda_{2,22} = \Phi_{2,22} + \frac{\omega}{1-\omega}\lambda_{1,22}$$

F-Test: General case of N entities and K variables

$$R \text{ vec } G \stackrel{!}{=} 0$$

$$R = \begin{bmatrix} \mathbf{I}_K \otimes R_1 & -\mathbf{I}_K \otimes R_2 & \mathbf{0}_{K^2 \times K^2 N} & \mathbf{0}_{K^2 \times K^2 N} & \cdots & \mathbf{0}_{K^2 \times K^2 N} & \mathbf{0}_{K^2 \times K^2 N} \\ \mathbf{0}_{K^2 \times K^2 N} & \mathbf{I}_K \otimes R_2 & -\mathbf{I}_K \otimes R_3 & \mathbf{0}_{K^2 \times K^2 N} & \cdots & \mathbf{0}_{K^2 \times K^2 N} & \mathbf{0}_{K^2 \times K^2 N} \\ \mathbf{0}_{K^2 \times K^2 N} & \mathbf{0}_{K^2 \times K^2 N} & \mathbf{I}_K \otimes R_3 & -\mathbf{I}_K \otimes R_4 & \cdots & \mathbf{0}_{K^2 \times K^2 N} & \mathbf{0}_{K^2 \times K^2 N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{K^2 \times K^2 N} & \mathbf{0}_{K^2 \times K^2 N} & \mathbf{0}_{K^2 \times K^2 N} & \mathbf{0}_{K^2 \times K^2 N} & \cdots & \mathbf{I}_K \otimes R_{N-1} & -\mathbf{I}_K \otimes R_N \end{bmatrix}_{K^2(N-1) \times K^2 N^2}$$

$$R_i = \begin{bmatrix} \frac{\omega_1}{\omega_i}\tau_{1i} & \frac{\omega_2}{\omega_i}\tau_{2i} & \cdots & \frac{\omega_N}{\omega_i}\tau_{Ni} \end{bmatrix}_{1 \times N} \otimes I_K$$

Proof of Proposition 1

Deriving the asymptotic estimated value of Ω:

$$\begin{aligned} \hat{\Omega} &= (\bar{X}'_{t-1}\bar{X}_{t-1})^{-1}\bar{X}'_{t-1}\bar{X}_t \\ &= (W'X'_{t-1}X_{t-1}W)^{-1}W'X'_{t-1}X_tW \\ &= (W'X'_{t-1}X_{t-1}W)^{-1}W'X'_{t-1}(X_{t-1}G + u_t)W \\ E[\hat{\Omega}] &= (W'X'_{t-1}X_{t-1}W)^{-1}W'X'_{t-1}X_{t-1}GW \\ &= (W'E[X'_{t-1}X_{t-1}]W)^{-1}W'E[X'_{t-1}X_{t-1}]GW \end{aligned}$$

$$E[X'_{t-1}X_{t-1}] = E[X'_tX_t] \equiv \Gamma_0$$

$$\begin{aligned}
E[X'_tX_t] &= E[(X_{t-1}G + u_t)'(X_{t-1}G + u_t)] \\
&= E[G'X'_{t-1}X_{t-1}G + G'X'_{t-1}u_t + u'_tX_{t-1}G + u'_tu_t] \\
&= G'E[X'_{t-1}X_{t-1}]G + u'_tu_t
\end{aligned}$$

Hence we have:

$$\Gamma_0 = G'\Gamma_0G + \Sigma$$

And finally:

$$E[\hat{\Omega}] = (W'\Gamma_0W)^{-1}W'\Gamma_0GW$$

Deriving the asymptotic estimated value of the shock $\hat{u}$:

$$\begin{aligned}
\hat{e}_t &= \bar{X}_t - \bar{X}_{t-1}\hat{\Omega} \\
&= X_tW - X_{t-1}W\hat{\Omega} \\
&= (X_{t-1}G + u_t)W - X_{t-1}W\hat{\Omega} \\
&= u_tW + X_{t-1}GW - X_{t-1}W\hat{\Omega} \\
&= u_tW + X_{t-1}(GW - W\hat{\Omega}) \\
&= u_tW + X_{t-1}(GW - W(W'\Gamma_0W)^{-1}W'\Gamma_0GW) \\
&= u_tW + X_{t-1}\underbrace{\left(I_{2\times 4} - W(W'\Gamma_0W)^{-1}W'\Gamma_0\right)GW}_{\Delta}
\end{aligned}$$

$$\bar{\Gamma}_0 \equiv E[\hat{e}'_t\hat{e}_t] = W'\Sigma W + \Delta'\Gamma_0\Delta$$

B. Tables and Figures

Table 1
GVAR of country pairs with two variables: Output and Inflation

	AT	BE	ES	FI	FR	DE	IR	IT	LU	NL	PT
AT	1.00	0.70	0.00	0.03	0.55	0.12	0.24	0.52	0.01	0.00	0.04
BE		1.00	0.00	0.02	0.07	0.47	0.25	0.95	0.01	0.00	0.06
ES			1.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
FI				1.00	0.23	0.80	0.05	0.19	0.00	0.04	0.27
FR					1.00	0.08	0.06	0.63	0.00	0.00	0.18
DE						1.00	0.00	0.00	0.00	0.99	0.19
IR							1.00	0.02	0.03	0.00	0.04
IT								1.00	0.00	0.01	0.54
LU									1.00	0.00	0.00
NL										1.00	0.08
PT											1.00

Note: The Table contains p-Values corresponding to the Wald test.

Table 2
GVAR of country pairs with two variables: Output and Government Spending

	AT	BE	ES	FI	FR	DE	IR	IT	LU	NL	PT
AT	1.00	0.10	0.00	0.65	0.05	0.28	0.00	0.00	0.01	0.45	0.00
BE		1.00	0.00	0.00	0.36	0.02	0.00	0.97	0.32	0.10	0.01
ES			1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
FI				1.00	0.05	0.67	0.00	0.89	0.25	0.00	0.00
FR					1.00	0.26	0.39	0.36	0.13	0.40	0.41
DE						1.00	0.03	0.68	0.17	0.22	0.36
IR							1.00	0.33	0.00	0.25	0.00
IT								1.00	0.27	0.82	0.15
LU									1.00	0.46	0.42
NL										1.00	0.28
PT											1.00

Note: The Table contains p-Values corresponding to the Wald test.

Table 3
Calibrated Trade Matrix and GDP weights:

Countries	AT	BE	GE	FI	FR	IR	IT	LU	NL	PO	ES
AT	0.0000	0.0343	0.6552	0.0106	0.0676	0.0064	0.1338	0.0031	0.0540	0.0051	0.0300
BE	0.0130	0.0000	0.2979	0.0097	0.2456	0.0448	0.0748	0.0228	0.2384	0.0101	0.0429
GE	0.1178	0.1417	0.0000	0.0223	0.2224	0.0229	0.1531	0.0110	0.2044	0.0190	0.0855
FI	0.0336	0.0823	0.3964	0.0000	0.1133	0.0229	0.1096	0.0027	0.1657	0.0141	0.0594
FR	0.0180	0.1735	0.3301	0.0094	0.0000	0.0199	0.1636	0.0109	0.1036	0.0222	0.1487
IR	0.0132	0.2482	0.2620	0.0147	0.1550	0.0000	0.0930	0.0021	0.1220	0.0177	0.0720
IT	0.0547	0.0805	0.3480	0.0140	0.2504	0.0184	0.0000	0.0054	0.0926	0.0160	0.1198
LU	0.0153	0.2968	0.3009	0.0041	0.2014	0.0048	0.0649	0.0000	0.0667	0.0143	0.0308
NL	0.0197	0.2296	0.4144	0.0188	0.1418	0.0216	0.0828	0.0049	0.0000	0.0132	0.0532
PO	0.0101	0.0519	0.2083	0.0087	0.1652	0.0170	0.0780	0.0054	0.0703	0.0000	0.3852
ES	0.0161	0.0607	0.2547	0.0100	0.2986	0.0185	0.1567	0.0034	0.0782	0.1032	0.0000
GDP weights	0.03	0.037	0.289	0.017	0.205	0.015	0.190	0.003	0.062	0.024	0.123

Figure 1
Simulations

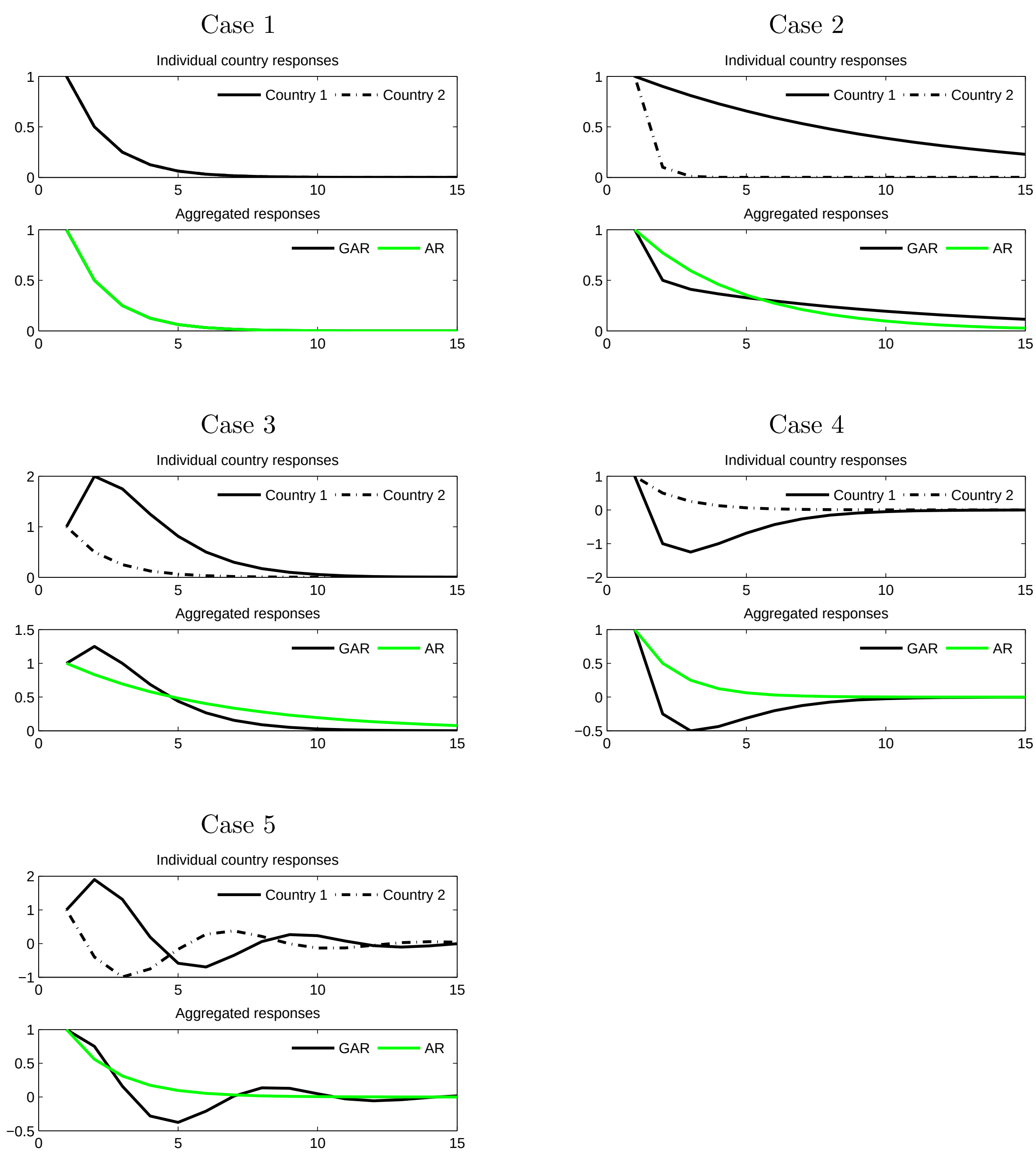


Figure 2
Comparison between GAR and AR processes.

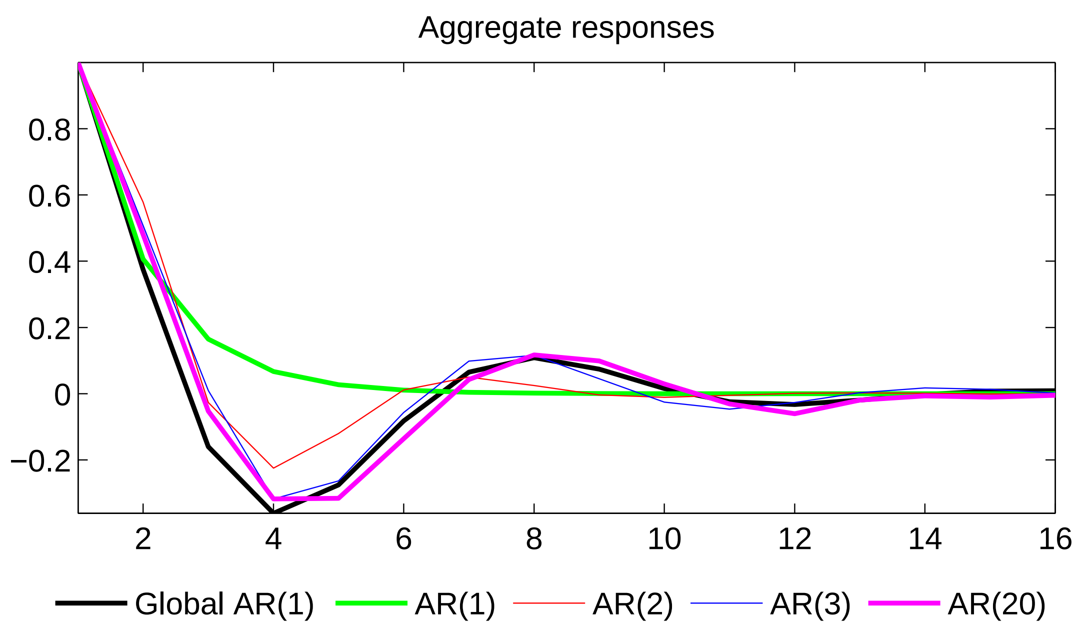


Figure 3
Spillovers across identical variables

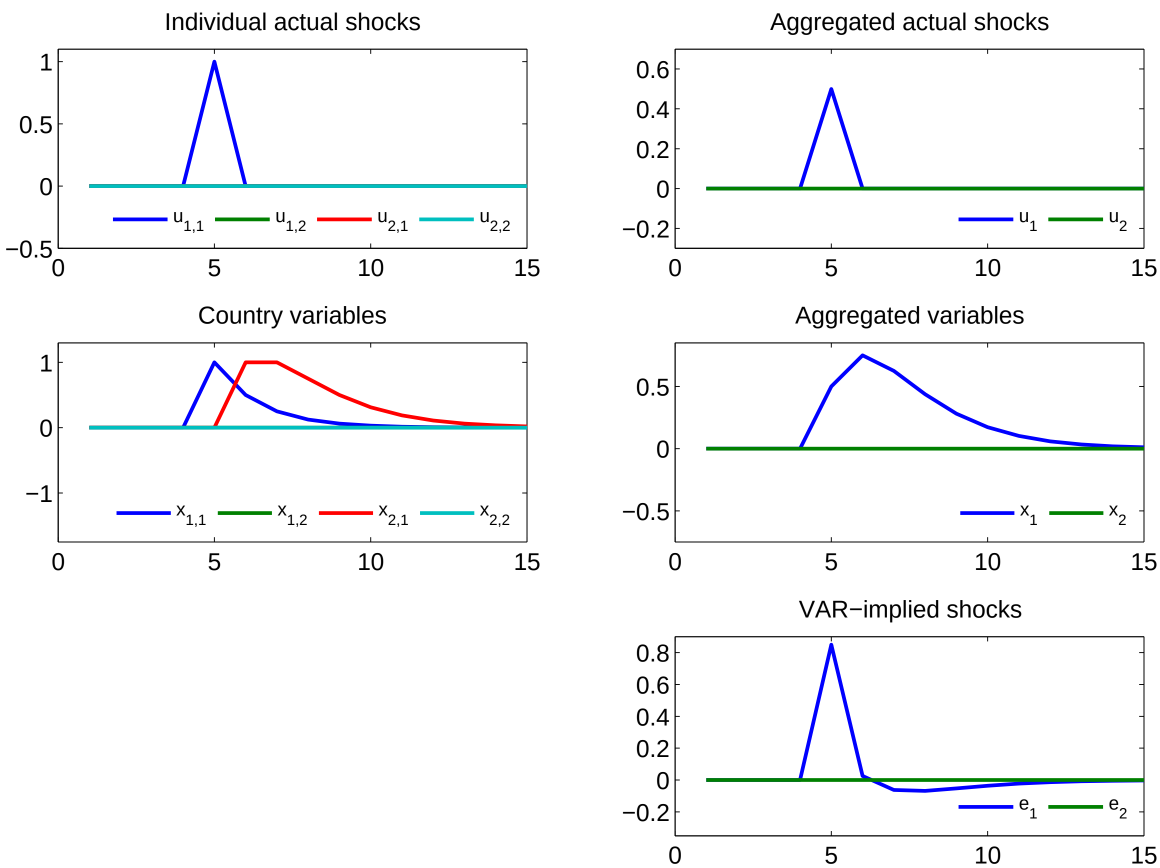


Figure 4
Spillovers across different variables

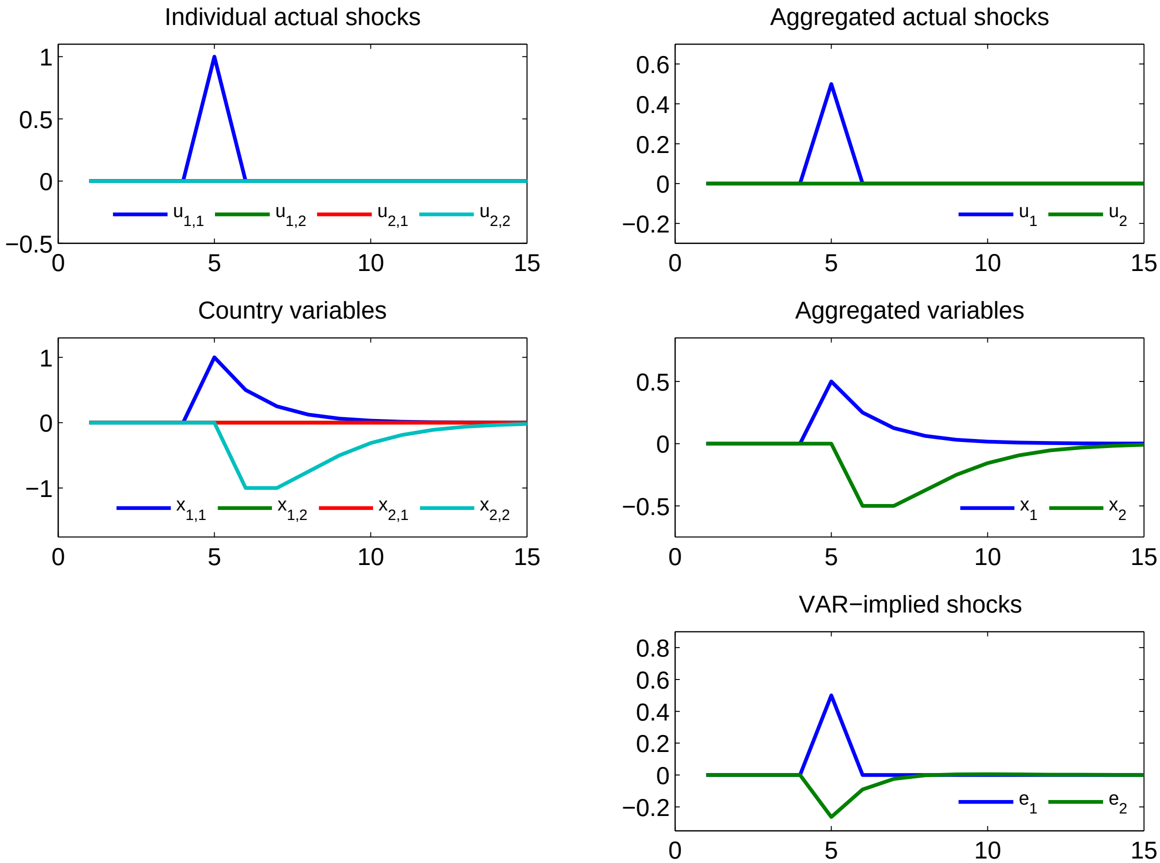
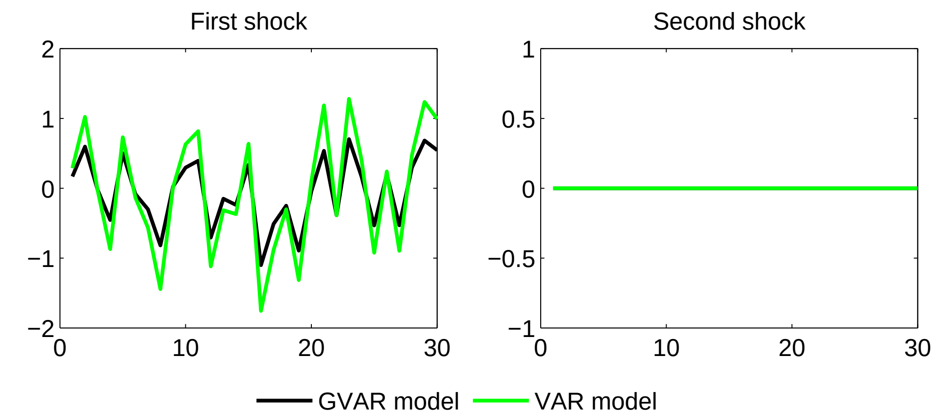
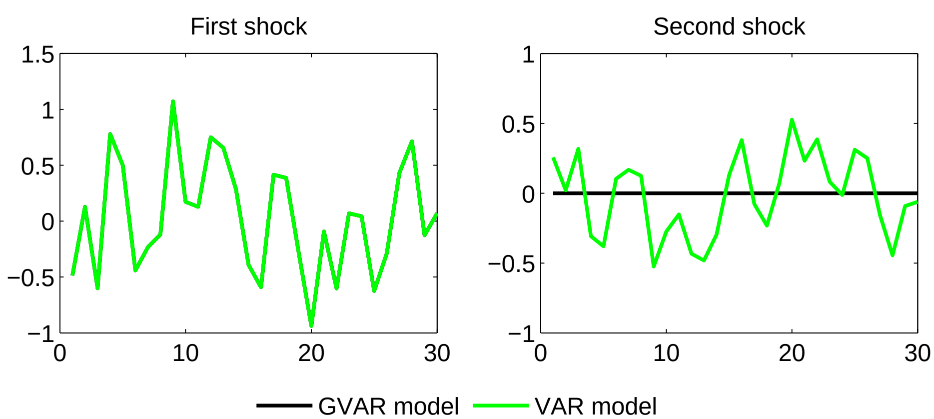


Figure 5

Spillovers across identical variables



Spillovers across different variables



Note: The Figure shows actual (GVAR) and fitted (VAR) time series of *aggregate* shocks. These time series are generated by simulating 30 time periods with only shocks to the first variable of the first country.

Figure 6

Variance Decomposition of VAR (left column) and GVAR (right column) 5 periods ahead

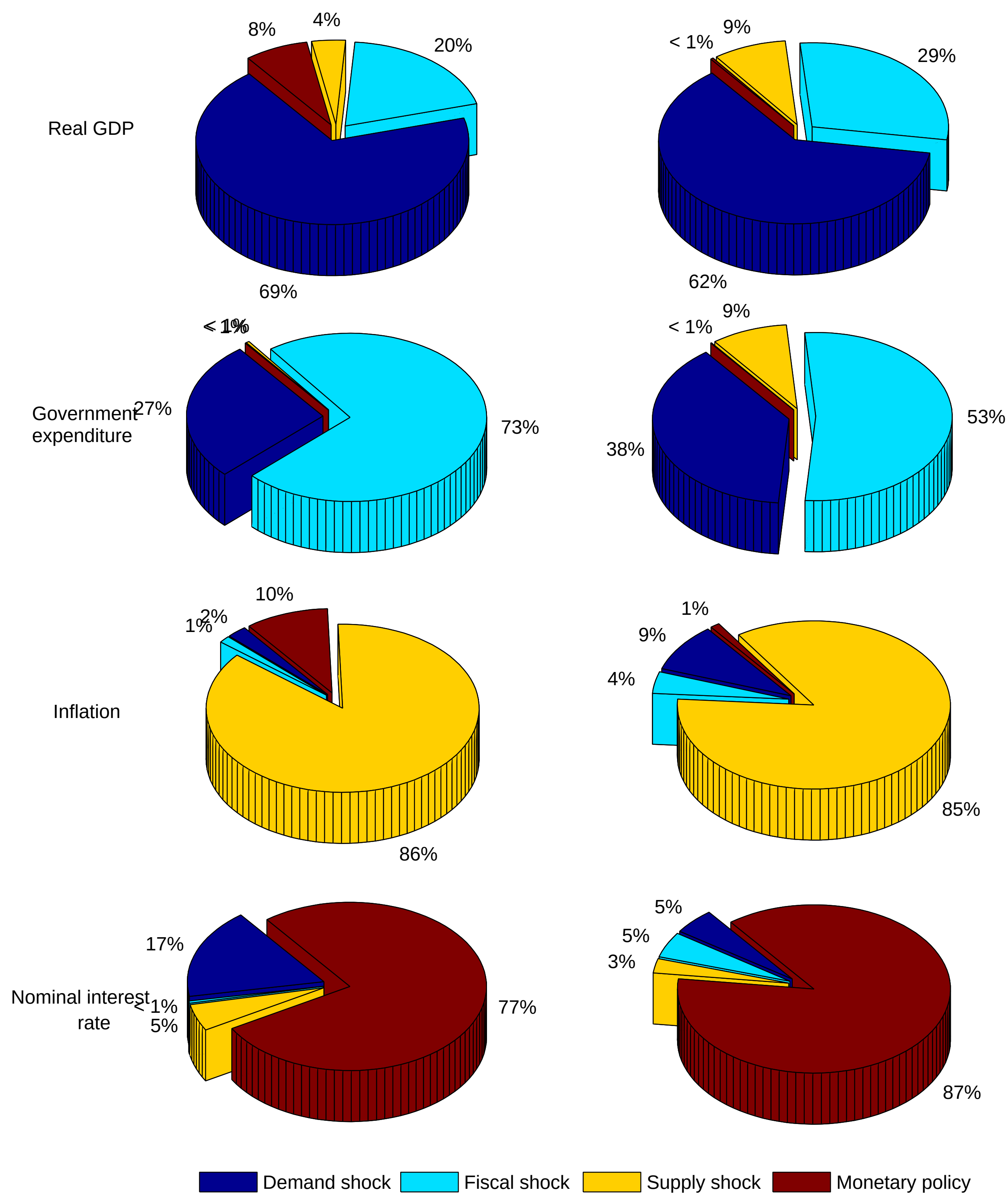


Figure 7
Variance Decomposition of VAR (left column) and GVAR (right column) 10 periods ahead

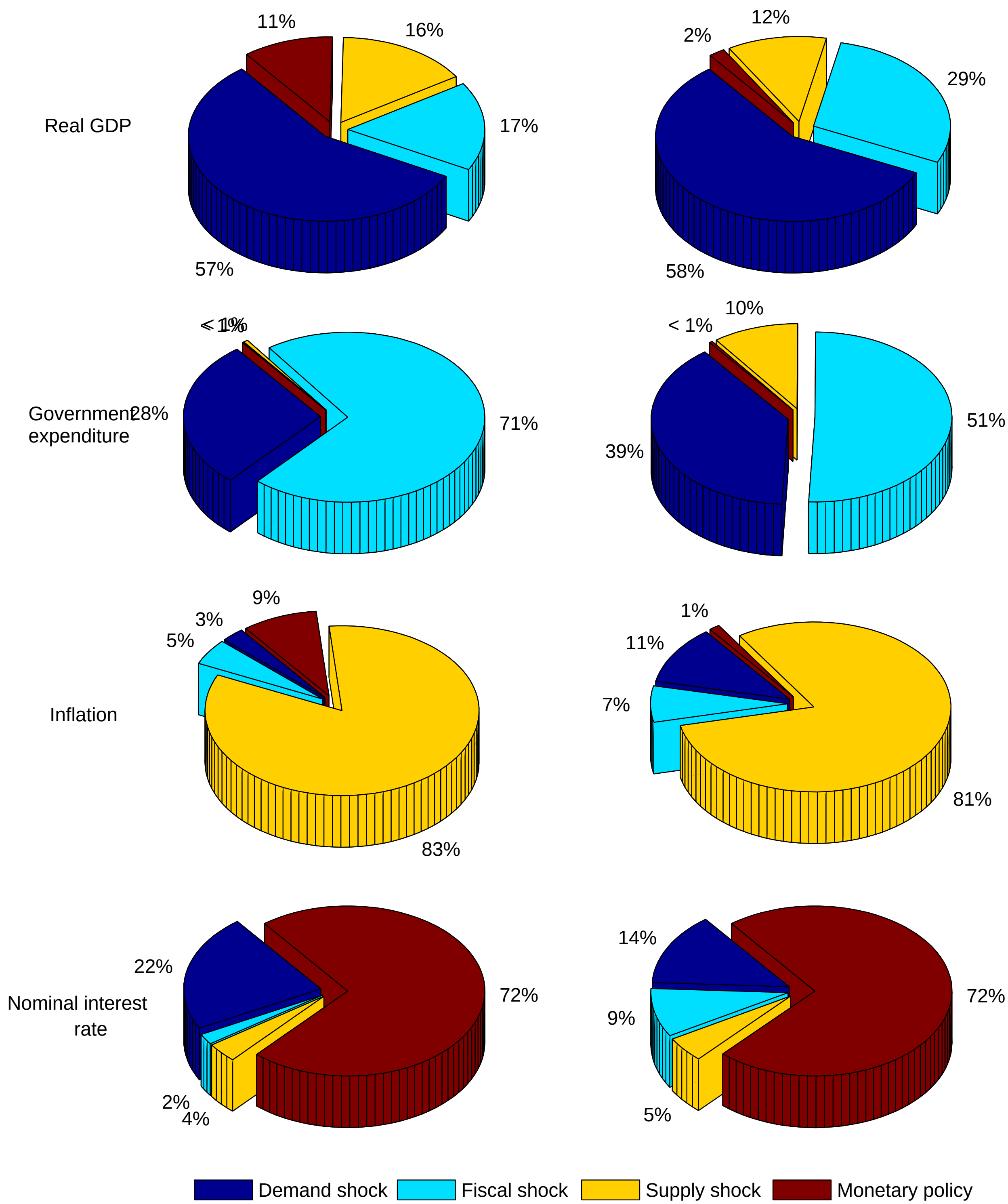


Figure 8

Variance Decomposition of VAR (left column) and GVAR (right column) 20 periods ahead

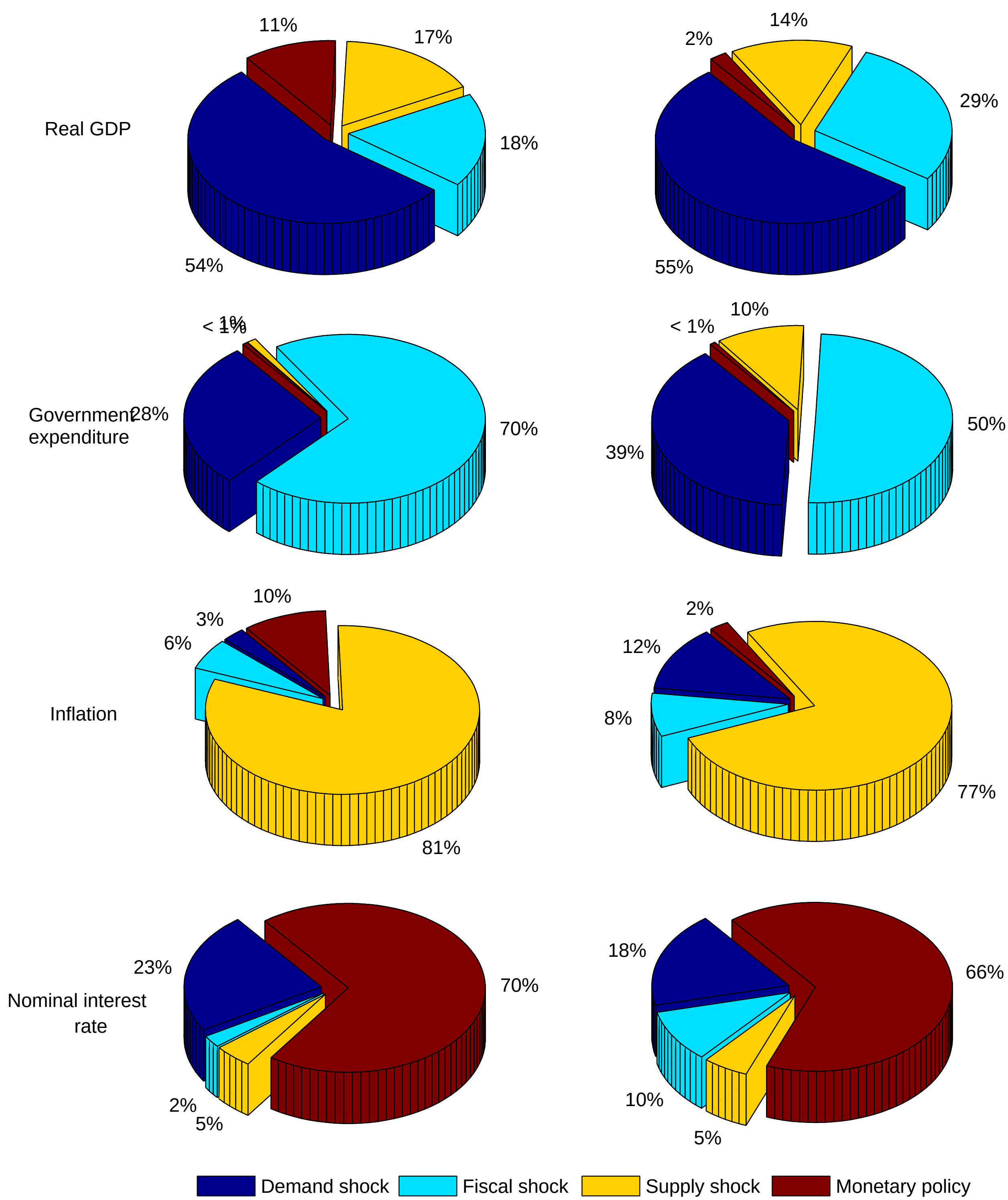


Figure 9

Variance Decomposition of VAR (left column) and GVAR (right column) 40 periods ahead

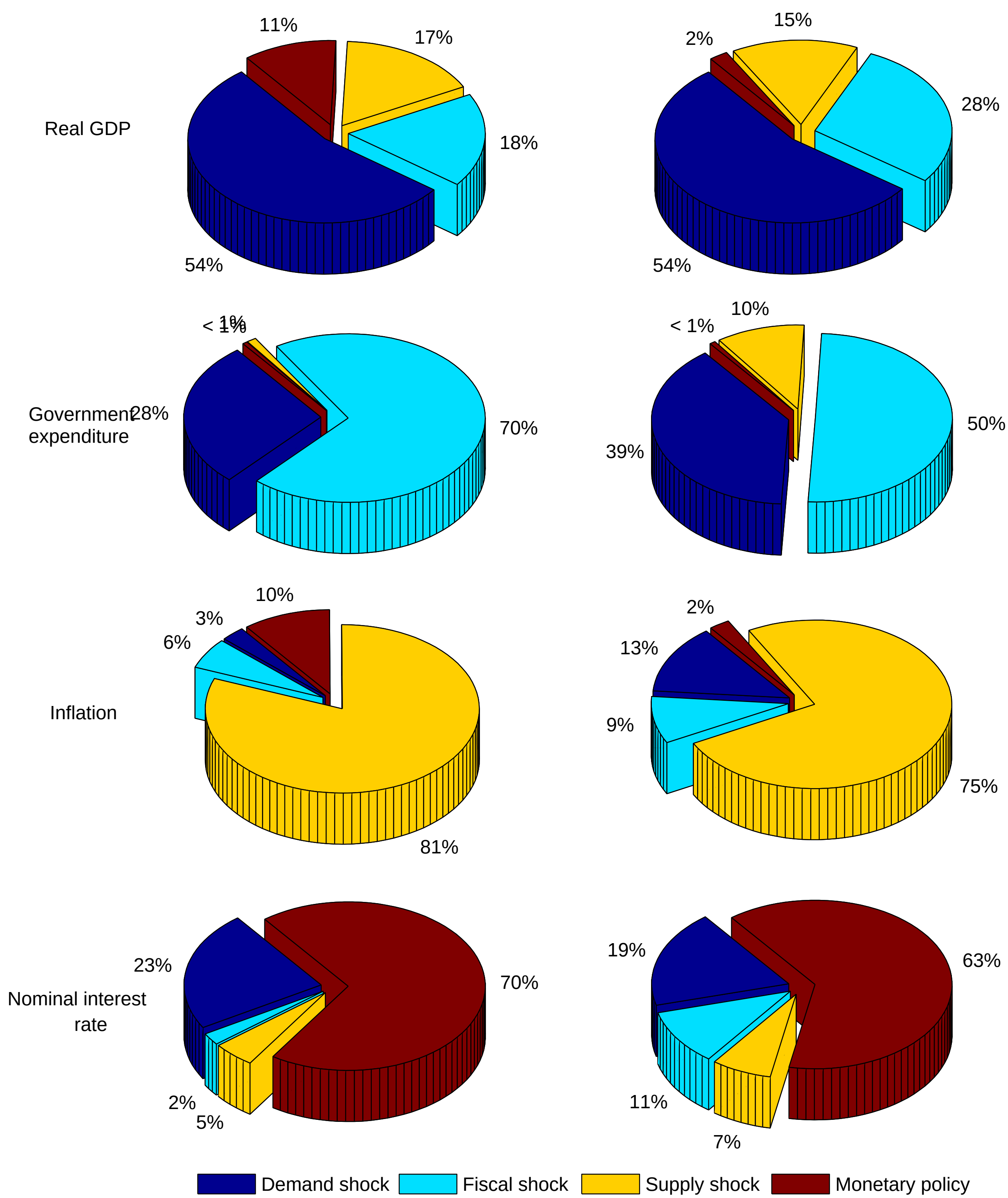


Figure 10
Historical Decomposition of VAR

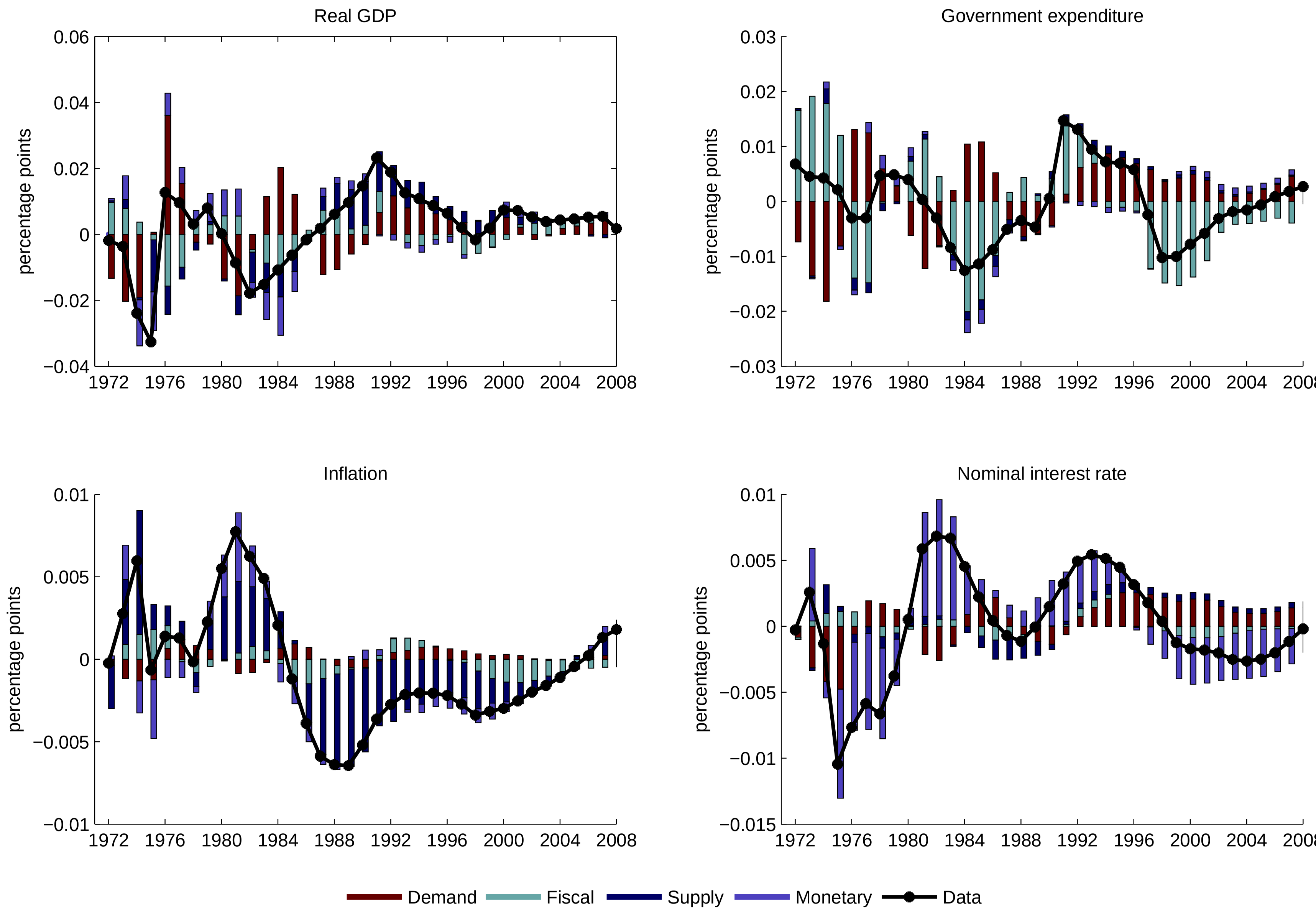


Figure 11
Historical Decomposition of GVAR