(G)VAR: When is the G essential?

Two Competing Views of Fluctuations and Shock Identification

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Abstract

In this paper the question is whether the VAR methodology is able to capture the dynamic and stochastic structure implied by the presence of spillovers in a GVAR system of interacting units. We derive analytical restrictions, propose a test indicating the degree to which aggregation is feasible and derive the asymptotic properties of the VAR estimator in the presence of spillovers. We illustrate the effects of misspecification through a set of numerical examples and find that failing to account for spillovers can lead the econometrician to mistakenly perceive higher persistence and volatility or recover shocks where there are none. Finally it is shown that an infinite order VAR can match the dynamics of a first order GVAR.

Roadmap

1. **Analytical results**
   - Dynamic equivalence
   - Asymptotic estimators
2. **Numerical results**
   - First example
   - Second example
   - More intuition

The Setup

Global Vector Autoregression model (GVAR) for a set of N countries:

$$\mathbf{x}_{i,t} = \mathbf{c}_i + \mathbf{d}_i t + \sum_{l=1}^{p} \Phi_{il}\mathbf{x}_{i,t-l} + \sum_{l=0}^{q} \Lambda_{il}\mathbf{x}^*_{i,t-l} + \sum_{l=0}^{r} \Psi_{il}\mathbf{f}_{t-l} + \mathbf{u}_{i,t}$$

$$\text{where } \mathbf{u}_{i,t} \sim N(0, \Sigma_i) \text{ and } \mathbf{x}^*_{i,t} = \sum_{j=1}^{N} \tau_{ij}\mathbf{x}_{j,t}$$

Nested version of the model:

· two countries
· two variables
· no intercept
· no trend
· no common factor
· one time lag
· no contemporaneous spillover

Two-country setup

Simplest two-country **GVAR** model:

$$\begin{bmatrix} x_{1,1t} \\ x_{1,2t} \end{bmatrix} = \begin{bmatrix} \Phi_{1,11} & \Phi_{1,12} \\ \Phi_{1,21} & \Phi_{1,22} \end{bmatrix} \begin{bmatrix} x_{1,1t-1} \\ x_{1,2t-1} \end{bmatrix} + \begin{bmatrix} \Lambda_{1,11} & \Lambda_{1,12} \\ \Lambda_{1,21} & \Lambda_{1,22} \end{bmatrix} \begin{bmatrix} x_{2,1t-1} \\ x_{2,2t-1} \end{bmatrix} + \begin{bmatrix} u_{1,1t} \\ u_{1,2t} \end{bmatrix}$$

$$\begin{bmatrix} x_{2,1t} \\ x_{2,2t} \end{bmatrix} = \begin{bmatrix} \Phi_{2,11} & \Phi_{2,12} \\ \Phi_{2,21} & \Phi_{2,22} \end{bmatrix} \begin{bmatrix} x_{2,1t-1} \\ x_{2,2t-1} \end{bmatrix} + \begin{bmatrix} \Lambda_{2,11} & \Lambda_{2,12} \\ \Lambda_{2,21} & \Lambda_{2,22} \end{bmatrix} \begin{bmatrix} x_{1,1t-1} \\ x_{1,2t-1} \end{bmatrix} + \begin{bmatrix} u_{2,1t} \\ u_{2,2t} \end{bmatrix}$$

Stacked representation of the model:

$$\begin{bmatrix} x_{1,1t} \\ x_{1,2t} \\ x_{2,1t} \\ x_{2,2t} \end{bmatrix} = \begin{bmatrix} \Phi_{1,11} & \Phi_{1,12} & \lambda_{1,11} & \lambda_{1,12} \\ \Phi_{1,21} & \Phi_{1,22} & \lambda_{1,21} & \lambda_{1,22} \\ \lambda_{2,11} & \lambda_{2,12} & \Phi_{2,11} & \Phi_{2,12} \\ \lambda_{2,21} & \lambda_{2,22} & \Phi_{2,21} & \Phi_{2,22} \end{bmatrix} \begin{bmatrix} x_{1,1t-1} \\ x_{1,2t-1} \\ x_{2,1t-1} \\ x_{2,2t-1} \end{bmatrix} + \begin{bmatrix} u_{1,1t-1} \\ u_{1,2t-1} \\ u_{2,1t-1} \\ u_{2,2t-1} \end{bmatrix}$$

Aggregated VAR

Define **aggregated** variables:

$$\bar{x}_{1,t} = \omega x_{1,1t} + (1-\omega) x_{2,1t}$$
$$\bar{x}_{2,t} = \omega x_{1,2t} + (1-\omega) x_{2,2t}$$

Define **aggregated** shocks:

$$\bar{u}_{1,t} = \omega u_{1,1t} + (1-\omega) u_{2,1t}$$
$$\bar{u}_{2,t} = \omega u_{1,2t} + (1-\omega) u_{2,2t}$$

VAR model:

$$\begin{bmatrix} \bar{x}_{1,t} \\ \bar{x}_{2,t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_{1,t-1} \\ \bar{x}_{2,t-1} \end{bmatrix} + \begin{bmatrix} \bar{u}_{1,t-1} \\ \bar{u}_{2,t-1} \end{bmatrix}$$

Dynamic equivalence

The models are **dynamically equivalent** if there exists a mapping M

$$\underbrace{\begin{bmatrix} \Phi_{1,11} & \Phi_{1,12} & \lambda_{1,11} & \lambda_{1,12} \\ \Phi_{1,21} & \Phi_{1,22} & \lambda_{1,21} & \lambda_{1,22} \\ \lambda_{2,11} & \lambda_{2,12} & \Phi_{2,11} & \Phi_{2,12} \\ \lambda_{2,21} & \lambda_{2,22} & \Phi_{2,21} & \Phi_{2,22} \end{bmatrix}}_{G} \xrightarrow{M} \underbrace{\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}}_{\Omega}$$

such that:

$$\bar{x}_{1,t|t-1} = \omega x_{1,1t|t-1} + (1-\omega) x_{2,1t|t-1}$$
$$\bar{x}_{2,t|t-1} = \omega x_{1,2t|t-1} + (1-\omega) x_{2,2t|t-1}$$

Econometric restrictions

VAR-implied dynamics:

$$\begin{aligned}\bar{x}_{1,t|t-1} &= \Omega_{11}\bar{x}_{1,t-1} + \Omega_{12}\bar{x}_{2,t-1} \\ &= \Omega_{11}[\omega x_{1,1t-1} + (1-\omega)x_{2,1t-1}] + \Omega_{12}[\omega x_{1,2t-1} + (1-\omega)x_{2,2t-1}] \\ &= \omega\Omega_{11}x_{1,1t-1} + (1-\omega)\Omega_{11}x_{2,1t-1} + \omega\Omega_{12}x_{1,2t-1} + (1-\omega)\Omega_{12}x_{2,2t-1}\end{aligned}$$

GVAR-implied dynamics:

$$\begin{aligned}&\omega x_{1,1t|t-1} + (1-\omega)x_{2,1t|t-1} = \\ &= [\omega\Phi_{1,11} + (1-\omega)\lambda_{2,11}]x_{1,1t-1} + [\omega\lambda_{1,11} + (1-\omega)\Phi_{2,11}]x_{2,1t-1} + \\ &\quad [\omega\Phi_{1,12} + (1-\omega)\lambda_{2,12}]x_{1,2t-1} + [\omega\lambda_{1,12} + (1-\omega)\Phi_{2,12}]x_{2,2t-1}\end{aligned}$$

Restrictions:

$$\Omega_{11} = \Phi_{1,11} + \frac{1-\omega}{\omega}\lambda_{2,11} \overset{!}{=} \Phi_{2,11} + \frac{\omega}{1-\omega}\lambda_{1,11}$$

$$\Omega_{12} = \Phi_{1,12} + \frac{1-\omega}{\omega}\lambda_{2,12} \overset{!}{=} \Phi_{2,12} + \frac{\omega}{1-\omega}\lambda_{1,12}$$

Econometric restrictions

VAR-implied dynamics:

$$\begin{aligned}\bar{x}_{1,t|t-1} &= \Omega_{11}\bar{x}_{1,t-1} + \Omega_{12}\bar{x}_{2,t-1} \\ &= \Omega_{11}[\omega x_{1,1t-1} + (1-\omega)x_{2,1t-1}] + \Omega_{12}[\omega x_{1,2t-1} + (1-\omega)x_{2,2t-1}] \\ &= \omega\Omega_{11}x_{1,1t-1} + (1-\omega)\Omega_{11}x_{2,1t-1} + \omega\Omega_{12}x_{1,2t-1} + (1-\omega)\Omega_{12}x_{2,2t-1}\end{aligned}$$

GVAR-implied dynamics:

$$\begin{aligned}&\omega x_{1,1t|t-1} + (1-\omega)x_{2,1t|t-1} = \\ &\quad = [\omega\Phi_{1,11} + (1-\omega)\lambda_{2,11}]x_{1,1t-1} + [\omega\lambda_{1,11} + (1-\omega)\Phi_{2,11}]x_{2,1t-1} + \\ &\quad [\omega\Phi_{1,12} + (1-\omega)\lambda_{2,12}]x_{1,2t-1} + [\omega\lambda_{1,12} + (1-\omega)\Phi_{2,12}]x_{2,2t-1}\end{aligned}$$

Restrictions:

$$\Omega_{11} = \Phi_{1,11} + \frac{1-\omega}{\omega}\lambda_{2,11} \stackrel{!}{=} \Phi_{2,11} + \frac{\omega}{1-\omega}\lambda_{1,11}$$

$$\Omega_{12} = \Phi_{1,12} + \frac{1-\omega}{\omega}\lambda_{2,12} \stackrel{!}{=} \Phi_{2,12} + \frac{\omega}{1-\omega}\lambda_{1,12}$$

Econometric restrictions

VAR-implied dynamics:

$$\begin{aligned}\bar{x}_{1,t|t-1} &= \Omega_{11}\bar{x}_{1,t-1} + \Omega_{12}\bar{x}_{2,t-1} \\ &= \Omega_{11}[\omega x_{1,1t-1} + (1-\omega)x_{2,1t-1}] + \Omega_{12}[\omega x_{1,2t-1} + (1-\omega)x_{2,2t-1}] \\ &= \omega\Omega_{11}x_{1,1t-1} + (1-\omega)\Omega_{11}x_{2,1t-1} + \omega\Omega_{12}x_{1,2t-1} + (1-\omega)\Omega_{12}x_{2,2t-1}\end{aligned}$$

GVAR-implied dynamics:

$$\begin{aligned}&\omega x_{1,1t|t-1} + (1-\omega)x_{2,1t|t-1} = \\ &= [\omega\Phi_{1,11} + (1-\omega)\lambda_{2,11}]x_{1,1t-1} + [\omega\lambda_{1,11} + (1-\omega)\Phi_{2,11}]x_{2,1t-1} + \\ &\quad [\omega\Phi_{1,12} + (1-\omega)\lambda_{2,12}]x_{1,2t-1} + [\omega\lambda_{1,12} + (1-\omega)\Phi_{2,12}]x_{2,2t-1}\end{aligned}$$

Restrictions:

$$\begin{aligned}\Omega_{11} &= \Phi_{1,11} + \frac{1-\omega}{\omega}\lambda_{2,11} \overset{!}{=} \Phi_{2,11} + \frac{\omega}{1-\omega}\lambda_{1,11} \\ \Omega_{12} &= \Phi_{1,12} + \frac{1-\omega}{\omega}\lambda_{2,12} \overset{!}{=} \Phi_{2,12} + \frac{\omega}{1-\omega}\lambda_{1,12}\end{aligned}$$

Econometric restrictions

Linear F-test:

$$R \text{ vec } G \overset{!}{=} 0$$

with:

$$R = \begin{bmatrix} R_1 & 0_{2\times 4} & R_2 & 0_{2\times 4} \\ 0_{2\times 4} & R_1 & 0_{2\times 4} & R_2 \end{bmatrix}$$

where:

$$R_1 = \begin{bmatrix} 1 & 0 & \frac{1-\omega}{\omega} & 0 \\ 0 & 1 & 0 & \frac{1-\omega}{\omega} \end{bmatrix} \text{ and } R_2 = \begin{bmatrix} -\frac{\omega}{1-\omega} & 0 & -1 & 0 \\ 0 & -\frac{\omega}{1-\omega} & 0 & -1 \end{bmatrix}$$

Natural extension: N countries, k variables

▸ Empirical Illustration

Effects of misspecification

Data generating process: GVAR

$$\mathbf{x}_t = G\mathbf{x}_{t-1} + \mathbf{u}_t, \qquad \mathbf{u}_t \sim N(0, \Sigma)$$

Aggregated VAR estimation:

$$\bar{\mathbf{x}}_t = \hat{\Omega}\bar{\mathbf{x}}_{t-1} + \hat{\mathbf{e}}_t, \qquad \hat{\mathbf{e}}_t \sim N(0, \hat{\Theta}) \qquad \text{with} \qquad \bar{\mathbf{x}}_t = W\mathbf{x}_t$$

Asymptotic estimators:

$$\hat{\Omega} = f(W, G, \Sigma) \text{ and } \hat{\Theta} = f(W, G, \Sigma)$$

Derivations

Asymptotic estimate of **coefficient matrix**:

$$\hat{\Omega} = (W'\Gamma_0 W)^{-1} W'\Gamma_0 G W, \text{ where } \Gamma_0 = G'\Gamma_0 G + \Sigma$$

Identification of **shocks**:

$$\begin{aligned} \hat{\mathbf{e}}_t &= \bar{\mathbf{x}}_t - \bar{\mathbf{x}}_{t-1}\hat{\Omega} \\ &= \mathbf{u}_t W + \bar{\mathbf{x}}_{t-1} \underbrace{\left(I_{2\times 4} - W(W'\Gamma_0 W)^{-1} W'\Gamma_0\right) GW}_{\Delta} \end{aligned}$$

Asymptotic estimate of **variance-covariance matrix**:

$$\hat{\Theta} = E[\hat{\mathbf{e}}_t'\hat{\mathbf{e}}_t] = \underbrace{W'\Sigma W}_{\Theta} + \Delta'\Gamma_0\Delta$$

$\Rightarrow$ **Stochastic** equivalence between Aggregated VAR and GVAR: $\Delta \overset{!}{=} 0$

Calibration

Stacked representation of the model:

$$\begin{bmatrix} x_{1,1t} \\ x_{1,2t} \\ x_{2,1t} \\ x_{2,2t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 1 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}}_{G} \begin{bmatrix} x_{1,1t-1} \\ x_{1,2t-1} \\ x_{2,1t-1} \\ x_{2,2t-1} \end{bmatrix} + \begin{bmatrix} u_{1,1t-1} \\ u_{1,2t-1} \\ u_{2,1t-1} \\ u_{2,2t-1} \end{bmatrix}$$

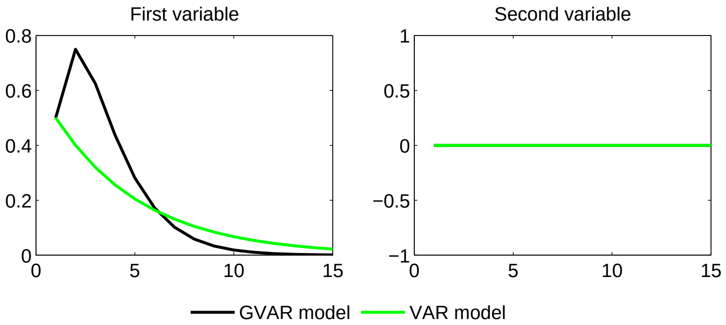
Stochastic structure:

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \omega = 0.5 \Rightarrow \Theta = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Asymptotic estimators:

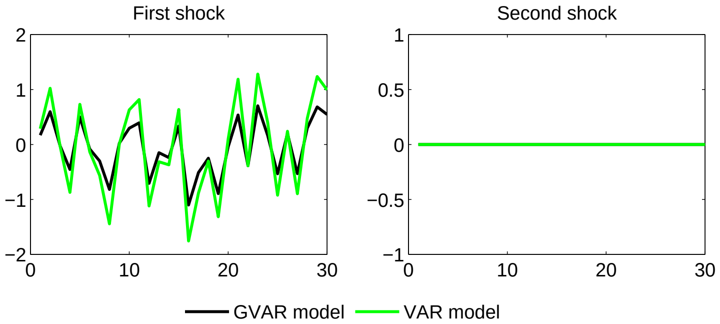
$$\hat{\Omega} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix} \text{ and } \hat{\Theta} = \begin{bmatrix} 0.6667 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Calibration



Note: The Figure shows impulse responses of aggregated variables $\bar{x}$ to a unitary shock to the first variable of the first country.

Calibration



Note: The Figure shows actual (GVAR) and fitted (VAR) time series of *aggregate* shocks. These time series are generated by simulating 30 time periods with ONLY SHOCKS TO THE FIRST VARIABLE OF THE FIRST COUNTRY.

Calibration

Stacked representation of the model:

$$\begin{bmatrix} x_{1,1t} \\ x_{1,2t} \\ x_{2,1t} \\ x_{2,2t} \end{bmatrix} = \underbrace{\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ -1 & 0 & 0 & 0.5 \end{bmatrix}}_{G} \begin{bmatrix} x_{1,1t-1} \\ x_{1,2t-1} \\ x_{2,1t-1} \\ x_{2,2t-1} \end{bmatrix} + \begin{bmatrix} u_{1,1t-1} \\ u_{1,2t-1} \\ u_{2,1t-1} \\ u_{2,2t-1} \end{bmatrix}$$

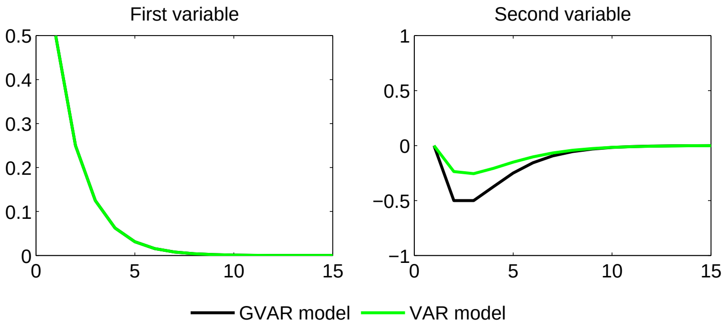
Stochastic structure:

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \omega = 0.5 \Rightarrow \Theta = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Asymptotic estimators:

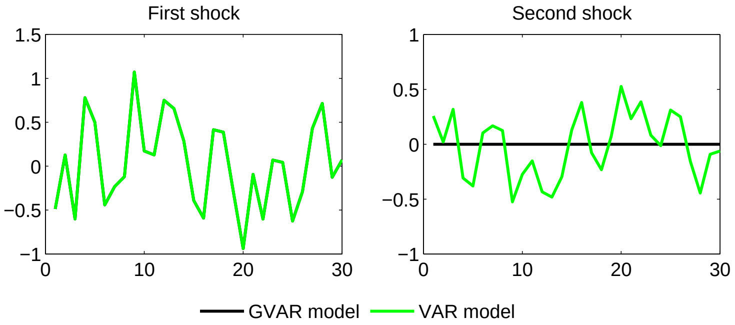
$$\hat{\Omega} = \begin{bmatrix} 0.5 & 0 \\ -0.4722 & 0.5833 \end{bmatrix} \text{ and } \hat{\Theta} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.6574 \end{bmatrix}$$

Calibration



Note: The Figure shows impulse responses of aggregated variables $\bar{x}$ to a unitary shock to the first variable in the first country.

Calibration



Note: The Figure shows actual (GVAR) and fitted (VAR) time series of *aggregate* shocks. These time series are generated by simulating 30 time periods with ONLY SHOCKS TO THE FIRST VARIABLE OF THE FIRST COUNTRY.

More intuition

We now consider a **univariate** setup.

Data generating process - Global AR(1) model:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} \phi_1 & \lambda_1 \\ \lambda_2 & \phi_2 \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

Aggregation:

$$\bar{x}_t = \omega x_{1,t} + (1-\omega) x_{2,t}$$

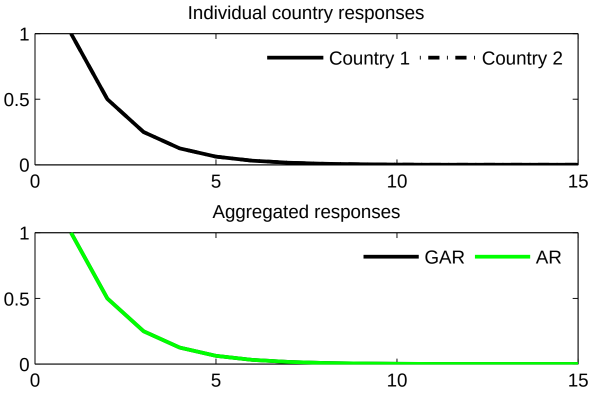
AR(1) estimation:

$$\bar{x}_t = \hat{\rho} \bar{x}_{t-1} + \hat{e}_t$$

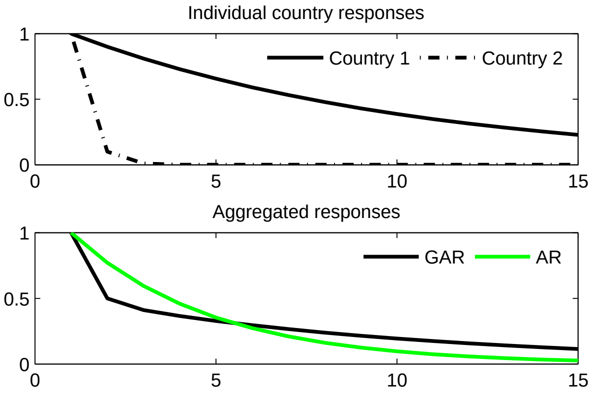
Calibrations: we consider 5 cases with $\omega = 0.5$.

Shocks: we analyze the effects of a unitary *aggregate* shock.

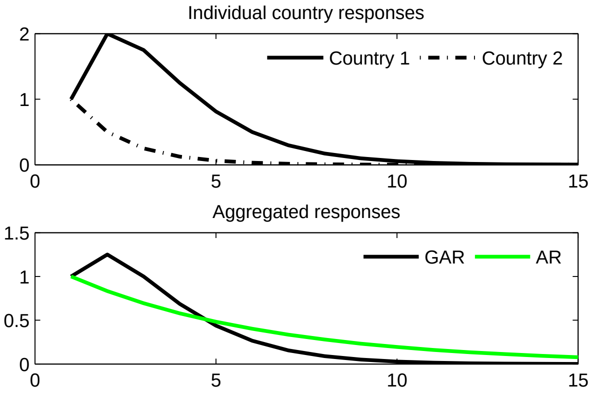
Calibration					**Estimation**
	ϕ_1	ϕ_2	λ_1	λ_2	$\hat{\rho}$
1:	0.5	0.5	0	0	0.5



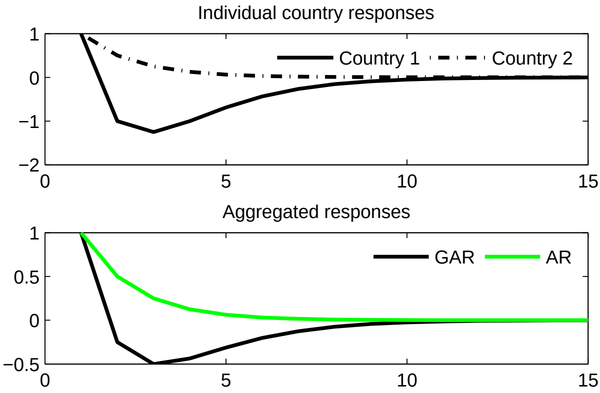
Calibration					**Estimation**
	ϕ_1	ϕ_2	λ_1	λ_2	$\hat{\rho}$
2:	0.9	0.1	0	0	0.77



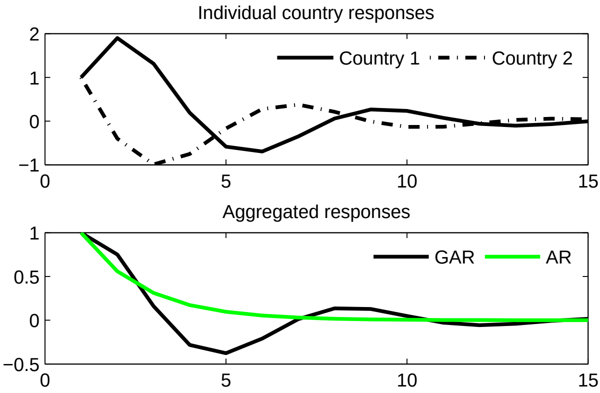
Calibration					Estimation
	ϕ_1	ϕ_2	λ_1	λ_2	$\hat{\rho}$
3:	0.5	0.5	1.5	0	0.83



Calibration					**Estimation**
	ϕ_1	ϕ_2	λ_1	λ_2	$\hat{\rho}$
4:	0.5	0.5	-1.5	0	0.5

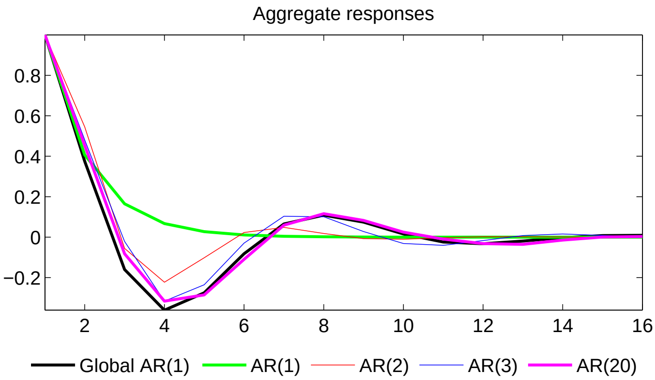


Calibration					**Estimation**
	ϕ_1	ϕ_2	λ_1	λ_2	$\hat{\rho}$
5:	0.9	0.1	1	-0.5	0.56



Calibration

	ϕ_1	ϕ_2	λ_1	λ_2
5':	0.8	0.2	0.5	-0.75



Conclusion

- It matters if one takes aggregated entities or disaggregated ones
- Therefore we propose a test in order to discriminate between the two alternatives
- Once N increases it becomes more and more improbable that the aggregation yields the same results as in the GVAR analysis
- In general it holds that with the lag length of the VAR going to infinity one can approximated the implied GVAR dynamics
- Doing simulations we find that not only the persistence can be different but also the direction of the impulse response function and the identification of shocks

Empirical Illustration I

Special case:

- pairwise **two-country** GVARs
- two variables: **inflation** and **output**

	AT	BE	ES	FI	FR	DE	IR	IT	LU	NL	PT
AT	1.00	0.70	0.00	0.03	0.55	0.12	0.24	0.52	0.01	0.00	0.04
BE		1.00	0.00	0.02	0.07	0.47	0.25	0.95	0.01	0.00	0.06
ES			1.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
FI				1.00	0.23	0.80	0.05	0.19	0.00	0.04	0.27
FR					1.00	0.08	0.06	0.63	0.00	0.00	0.18
DE						1.00	0.00	0.00	0.00	0.99	0.19
IR							1.00	0.02	0.03	0.00	0.04
IT								1.00	0.00	0.01	0.54
LU									1.00	0.00	0.00
NL										1.00	0.08
PT											1.00

Note: The Table contains p-Values corresponding to the Wald test.

Empirical Illustration II

Special case:

- pairwise **two-country** GVARs
- two variables: **inflation** and **government spending**

	AT	BE	ES	FI	FR	DE	IR	IT	LU	NL	PT
AT	1.00	0.10	0.00	0.65	0.05	0.28	0.00	0.00	0.01	0.45	0.00
BE		1.00	0.00	0.00	0.36	0.02	0.00	0.97	0.32	0.10	0.01
ES			1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
FI				1.00	0.05	0.67	0.00	0.89	0.25	0.00	0.00
FR					1.00	0.26	0.39	0.36	0.13	0.40	0.41
DE						1.00	0.03	0.68	0.17	0.22	0.36
IR							1.00	0.33	0.00	0.25	0.00
IT								1.00	0.27	0.82	0.15
LU									1.00	0.46	0.42
NL										1.00	0.28
PT											1.00

Note: The Table contains p-Values corresponding to the Wald test.

▸ Econometric restrictions

Full Derivations I

What is the asymptotic estimated value of Ω?

$$\begin{aligned}\hat{\Omega} &= (\bar{X}'_{t-1}\bar{X}_{t-1})^{-1}\bar{X}'_{t-1}\bar{X}_t \\ &= (W'X'_{t-1}X_{t-1}W)^{-1}W'X'_{t-1}X_tW \\ &= (W'X'_{t-1}X_{t-1}W)^{-1}W'X'_{t-1}(X_{t-1}G + u_t)W \\ E[\hat{\Omega}] &= (W'X'_{t-1}X_{t-1}W)^{-1}W'X'_{t-1}X_{t-1}GW \\ &= (W'E[X'_{t-1}X_{t-1}]W)^{-1}W'E[X'_{t-1}X_{t-1}]GW\end{aligned}$$

$$E[X'_{t-1}X_{t-1}] = E[X'_tX_t] \equiv \Gamma_0$$

$$\begin{aligned}E[X'_tX_t] &= E[(X_{t-1}G + u_t)'(X_{t-1}G + u_t)] \\ &= E[G'X'_{t-1}X_{t-1}G + G'X'_{t-1}u_t + u'_tX_{t-1}G + u'_tu_t] \\ &= G'E[X'_{t-1}X_{t-1}]G + u'_tu_t\end{aligned}$$

Hence we have:

$$\Gamma_0 = G'\Gamma_0G + \Sigma$$

And finally:

$$E[\hat{\Omega}] = (W'\Gamma_0W)^{-1}W'\Gamma_0GW$$

Full Derivations II

What is the asymptotic estimated value of the shock u

$$\begin{aligned}
\hat{e}_t &= \bar{X}_t - \bar{X}_{t-1}\hat{\Omega} \\
&= X_t W - X_{t-1} W\hat{\Omega} \\
&= (X_{t-1}G + u_t)W - X_{t-1}W\hat{\Omega} \\
&= u_t W + X_{t-1}GW - X_{t-1}W\hat{\Omega} \\
&= u_t W + X_{t-1}(GW - W\hat{\Omega}) \\
&= u_t W + X_{t-1}(GW - W(W'\Gamma_0 W)^{-1}W'\Gamma_0 GW) \\
&= u_t W + X_{t-1} \underbrace{\left(I_{2\times 4} - W(W'\Gamma_0 W)^{-1}W'\Gamma_0\right) GW}_{\Delta}
\end{aligned}$$

$$\bar{\Gamma}_0 \equiv E[\hat{e}_t'\hat{e}_t] = W'\Sigma W + \Delta'\Gamma_0\Delta$$